Describing Oscillations of High Energy Neutrinos in Matter Precisely

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We present a formalism for precise description of oscillation phenomena in matter at high energies or high densities, $V > \Delta m^2/2E$, where V is the matter-induced potential of neutrinos. The accuracy of the approximation is determined by the quantity $\sin^2 2\theta_m \Delta V/2\pi V$, where θ_m is the mixing angle in matter and ΔV is a typical change of the potential over the oscillation length ($l \sim 2\pi/V$). We derive simple and physically transparent formulas for the oscillation probabilities, which are valid for arbitrary matter density profiles. They can be applied to oscillations of high-energy ($E > 10$ GeV) accelerator, atmospheric, and cosmic neutrinos in the matter of the Earth, substantially simplifying numerical calculations and providing an insight into the physics of neutrino oscillations in matter. The effect of parametric enhancement of the oscillations of high-energy neutrinos is considered.

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Introduction.—Neutrino physics enters a new phase now, where the objectives are precision measurements of the parameters, studies of subleading oscillation effects, and searches for new physics beyond the already standard picture, which includes nonzero neutrino masses and mixing. Detection of neutrinos from new sources, in particular, of cosmic neutrinos, is in the agenda.

Substantial new information is expected from the studies of high-energy ($E > 1$ GeV) neutrinos. This includes investigations of atmospheric neutrinos with new large volume detectors [1], long baseline accelerator experiments [2], and detection of cosmic neutrinos from galactic and extragalactic sources [3]. Another possible source of neutrinos is annihilation of hypothetical weakly interacting massive particles (WIMPs) in the center of the Earth and the Sun [4]. In all these cases beams of high-energy neutrinos can propagate significant distances in the matter of the Earth (or of the Sun) and therefore undergo oscillations or conversions in matter.

Increased accuracy and reach of neutrino experiments put forward new and more challenging demands to the theoretical description of neutrino oscillations. In the present Letter our primary goal is to study oscillations of high-energy neutrinos [5], but the formulas we obtain are actually applicable in a wide range of neutrino energies. They simplify substantially numerical calculations and allow a deep insight into the physics of neutrino conversions in matter. In particular, they provide a useful tool for studying parametric enhancement of neutrino oscillations. The parametric enhancement occurs when the variation of the matter density along the neutrino trajectory is in a certain way correlated with the change of the oscillation phase [6,7].

Formalism.—We consider oscillations in the 3-flavor neutrino system (ν_e, ν_μ, ν_τ), with the mass squared differences Δm_{31}^2 and Δm_{21}^2 responsible for the oscillations of atmospheric and solar neutrinos, respectively. We shall be mainly interested in oscillations of neutrinos with energies $E > \Delta m_{31}^2/2V$, where the matter-induced potential of neutrinos $V(x) \equiv \sqrt{2} G_F N_e(x)$, with $N_e(x)$ the electron number density in matter and G_F the Fermi constant [8,9]. This corresponds to $E > 8$–10 GeV for the matter of the Earth. In this case the 1–2 mixing is strongly suppressed by matter, and the problem is reduced to an effective two-flavor one, described by the mass squared difference $\Delta m^2 \equiv \Delta m_{31}^2$ and mixing angle $\theta \equiv \theta_{13}$ (which is assumed to be nonzero) [10]. In particular, the oscillations of electron neutrinos are determined by the transition probability $P_2 \equiv P(\nu_e \leftrightarrow \nu_a)$, where $\nu_a = \sin\theta_{23}\nu_\mu + \cos\theta_{23}\nu_\tau$. In terms of P_2 the flavor transition probabilities are $P(\nu_e \leftrightarrow \nu_\mu) = \sin^2\theta_{23} P_2$, $P(\nu_e \leftrightarrow \nu_\tau) = \cos^2\theta_{23} P_2$ [10].

In the (ν_e, ν_a) basis, the evolution matrix $S(x)$ describing neutrino oscillations satisfies the equation

$$i\frac{dS}{dx} = H(x)S, \tag{1}$$

with the Hamiltonian

$$H(x) = \frac{V}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \delta\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{2}$$

Here $\delta \equiv \Delta m^2/4E$, and the first (potential) term dominates in the high-energy limit. However, in most situations of interest the neutrino path length in matter L satisfies $\delta \cdot L \gtrsim 1$; therefore, we cannot consider the whole second term as a small perturbation, and the effect of δ on the neutrino energy level splitting should be taken into account. For this reason we decompose the Hamiltonian as $H = H_0 + H_I$ with

$$H_0 = \omega\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_I = \sin 2\theta\delta\begin{pmatrix} -\epsilon & 1 \\ 1 & \epsilon \end{pmatrix}. \quad (3)$$

Here

$$\omega(x) \equiv \sqrt{(V/2 - \delta\cos 2\theta)^2 + \delta^2\sin^2 2\theta}, \quad (4)$$

2ω being the difference of the eigenvalues of $H(x)$;

$$\epsilon \equiv \frac{\cos 2\theta\delta - V/2 + \omega}{\sin 2\theta\delta} \approx \frac{\delta}{V}\sin 2\theta \ll 1. \quad (5)$$

The ratio of the second and the first terms in the Hamiltonian (3) is given by the mixing angle in matter θ_m: $\sin 2\theta\delta/\omega = \sin 2\theta_m$. Therefore, for $\sin 2\theta_m \ll 1$ the term H_I can be considered as a perturbation. Furthermore, according to (5), $\epsilon \sim \sin 2\theta_m$, so that the diagonal terms in H_I can be neglected in the lowest approximation.

We seek the solution of Eq. (1) in the form $S = S_0 \cdot S_I$, where S_0 is the solution of the evolution equation with H replaced by H_0. From (3) we find

$$S_0(x) = \begin{pmatrix} e^{-i\phi(x)} & 0 \\ 0 & e^{i\phi(x)} \end{pmatrix}, \quad (6)$$

where

$$\phi(x) \equiv \int_0^x dx'\omega(x') \quad (7)$$

is the adiabatic phase. Then, according to (1), the matrix S_I satisfies the equation

$$i\frac{dS_I}{dx} = S_0^{-1}H_I S_0 S_I = \tilde{H}_I S_I, \quad (8)$$

where $\tilde{H}_I \equiv S_0^{-1}H_I S_0$ is the perturbation Hamiltonian in the "interaction" representation. Equation (8) can be solved by iterations: $S_I = \mathbb{1} + S_I^{(1)} + \ldots$, which leads to the standard perturbation series for the S matrix. For neutrino propagation between $x = 0$ and $x = L$ we have, to the lowest nontrivial order,

$$S(L) = S_0(L)\left[\mathbb{1} - i\delta\sin 2\theta\int_0^L dx\begin{pmatrix} 0 & e^{i2\phi(x)} \\ e^{-i2\phi(x)} & 0 \end{pmatrix}\right]. \quad (9)$$

The $\nu_e \leftrightarrow \nu_a$ transition probability P_2 is given by the squared modulus of the off-diagonal element $[S(L)]_{ae}$:

$$P_2 = \delta^2\sin^2 2\theta\left|\int_0^L dx e^{-i2\phi(x)}\right|^2. \quad (10)$$

For density profiles that are symmetric with respect to the center of the neutrino trajectory, $V(x) = V(L - x)$, Eq. (10) gives

$$P_2 = 4\left(\frac{\Delta m^2}{4E}\right)^2\sin^2 2\theta\left[\int_0^{L/2} dz\cos 2\phi(z)\right]^2, \quad (11)$$

where $z = x - L/2$ is the distance from the midpoint of the trajectory and $\phi(z)$ is the phase acquired between this midpoint and the point z.

The transition probability P_2 scales with neutrino energy essentially as E^{-2}. The accuracy of Eq. (10) also improves with energy as E^{-2}. This is illustrated by Figs. 1(a) and 1(b). One can see that already for $E \gtrsim 8$ MeV the accuracy of our analytic formula is extremely good. Note that when neutrinos do not cross the Earth's core ($\cos\Theta > -0.837$, where Θ is the zenith angle of the neutrino trajectory) and so experience a slowly changing potential $V(x)$, the accuracy of the approximation (10) is very good even in the Mikheev-Smirnov-Wolfenstein (MSW) resonance region $E \sim 5$–8 GeV. The accuracy of Eq. (10) is also good for energies below ~ 2 GeV (not shown in the figure); however, in this region the domain of the applicability of (10) is relatively narrow, since for $E \lesssim 0.5$ GeV the oscillations driven by the "solar" parameters (Δm_{21}^2, θ_{12}) can no longer be neglected.

To understand the remarkable accuracy of Eq. (10), we find the correction ΔP_2 to the transition probability in (10) emerging in the next nontrivial order in H_I. Note that from the above considerations one can expect $\Delta P_2/P_2$ to be proportional to $\sin 2\theta_m$. Furthermore, for uniform matter Eq. (10) reproduces the exact transition probability; therefore, one expects $\Delta P_2/P_2 \propto V'$. A straightforward calculation indeed gives $\Delta P_2/P_2 \simeq \sin^2 2\theta_m(\Delta V/4\pi\omega) \simeq \sin^2 2\theta_m(\Delta V/2\pi V)$, where ΔV is the change of the potential over the oscillation length π/ω, and the last equality holds in the high-energy regime. For slowly changing density this is equivalent to $\Delta P_2/P_2 \simeq \sin^2 2\theta_m(V'/4\omega^2)$. Introducing the adiabaticity parameter $\gamma = 4\pi\omega/(\sin 2\theta_m\Delta V)$, we find that $\Delta P_2/P_2 \simeq \sin 2\theta_m\gamma^{-1}$, and therefore for small mixing in matter our approximation is better than the adiabatic one. At the same time, for $\Delta V/4\pi\omega < 1$ it is better than the simple expansion in powers of $\sin^2 2\theta_m$.

The matter density profile of the Earth satisfies $V'/V^2 \lesssim 0.5$, and therefore for oscillations in the Earth our approximation is expected to work well when $\sin^2 2\theta_m \ll 1$. This is fulfilled in the high-energy (or, equivalently, high-density) limit $EV/\Delta m^2 \gg \cos 2\theta$, i.e., above the MSW resonance. If the vacuum mixing angle is small ($\theta = \theta_{13}$), our expansion parameter is also small below the resonance. The above formalism applies in this low energy case as well, with only minor modifications: the sign of H_0 in (3) has to be flipped, and correspondingly one has to replace $\omega \to -\omega$ in Eq. (5). Expressions for the transition probability in Eq. (10) and (11) remain unchanged. Thus, our results are in general valid outside the MSW resonance region, which for small θ is very narrow. For the non-resonant channels ($\bar{\nu}$ channels for $\Delta m^2 > 0$ or ν channels for $\Delta m^2 < 0$) and small vacuum mixing, our formulas are valid in the whole diapason of energies because $\sin 2\theta_m$ is always small.

If θ_{13} is very small or vanishes, $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations are driven by $\Delta m^2 = \Delta m_{21}^2$ and the large mixing angle $\theta = \theta_{12}$. The oscillation probabilities can then be expressed through another effective two-flavor probability,

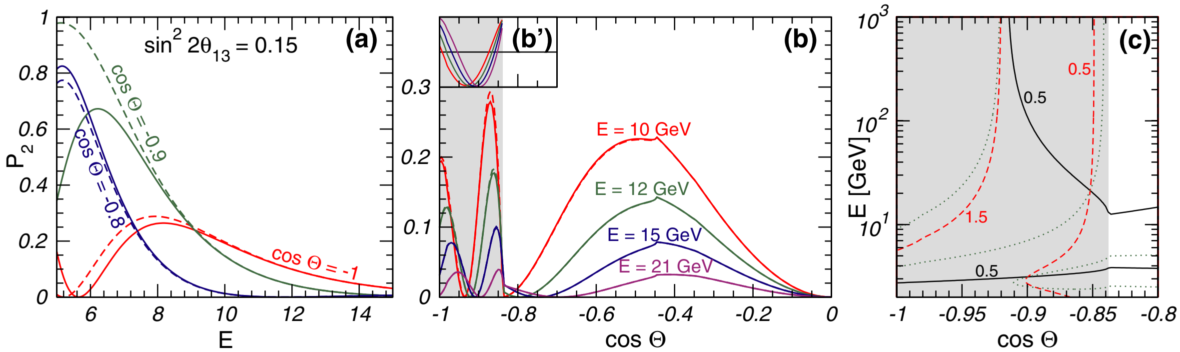

FIG. 1 (color online). Transition probability P_2 vs neutrino energy E for different trajectories inside the Earth [panel (a)] and vs the cosine of the zenith angle of the neutrino trajectory, $\cos\Theta$, for different neutrino energies [panel (b)]. Solid curves are the results of exact numerical calculations; dashed curves are obtained using formula (10). Small window (b') in panel (b) shows the values of the parameter X_3 calculated in the three-layer model of the Earth's density. Panel (c): contours of constant phases ϕ_m (solid line), ϕ_c (dashed line), and $X_3 = 0$ (dotted line). The numbers at the curves are the values of the phases in units of π. The shaded areas in panels (b) and (c) correspond to the Earth's core. For all panels we take $\sin^2 2\theta_{13} = 0.15$ and $\Delta m^2 = 2 \times 10^{-3}$ eV^2.

$\tilde{P}_2 \equiv \tilde{P}_2[\Delta m_{21}^2, \theta_{12}, V(x)]$, in terms of which $P(\nu_e \leftrightarrow \nu_\mu) = \cos^2\theta_{23}\tilde{P}_2$, $P(\nu_e \leftrightarrow \nu_\tau) = \sin^2\theta_{23}\tilde{P}_2$ [11]. For $EV/\Delta m_{21}^2 \gg \cos 2\theta_{12}$ (which for the typical densities inside the Earth corresponds to $E \gtrsim 0.5$ GeV), the probability $\tilde{P}_2$ is very well approximated by Eq. (10).

Let us consider the case of symmetric matter density profiles. Integrating (11) by parts, one finds

$$P_2 = \sin^2 2\theta_m^0 \left[\sin\phi_L + \omega_0 \int_0^{L/2} dz \frac{d\omega}{dz} \frac{1}{\omega^2} \sin 2\phi(z) \right]^2, \tag{12}$$

where $\theta_m^0 \equiv \theta_m(V_0)$, $\omega_0 \equiv \omega(V_0)$, V_0 being the potential at the initial and final points of the neutrino trajectory, and ϕ_L is the adiabatic phase acquired along the entire neutrino path. If the potential changes slowly with distance, so that $\omega^{-2} d\omega/dz \ll 1$, the second term in (12) can be neglected, and P_2 reduces to the usual adiabatic probability in symmetric matter: $P_{\rm adiab} = \sin^2 2\theta_m^0 \sin^2\phi_L$. The second term in (12) describes the effects of violation of adiabaticity.

Let us apply the above results to neutrino beams crossing the Earth. According to the preliminary reference earth model (PREM) [12], the Earth density profile can be described as several spherical shells of radii R_i with smooth density change within the shells and sharp change at the borders between them. Then, along a direction from the center of the Earth outwards, $\omega(z)$ decreases abruptly from ω_i^+ to ω_i^- in very narrow regions around R_i. Therefore, $d\omega/dz$ is large in these narrow regions and small outside them. The integration in (12) can then be easily done, leading to

$$P_2 \approx \sin^2 2\theta_m^0 \left[\sin\phi_L - \omega_0 \Sigma_i \frac{\omega_i^+ - \omega_i^-}{\omega_i^+ \omega_i^-} \sin 2\phi_i \right]^2. \tag{13}$$

Here ϕ_i is the adiabatic phase acquired by neutrinos between the points $z = 0$ and $z = R_i$.

Parametric enhancement of oscillations.—Inside the Earth, all the density jumps between different shells, except those between the mantle and core are relatively small [12]. Therefore, the density profile felt by neutrinos crossing the core of the Earth can be approximated by three layers (mantle-core-mantle). Equation (13) then gives

$$P_2 \approx \sin^2 2\theta_m^0 \left[\sin(\phi_c + 2\phi_m) - \frac{\omega_0}{\omega_m} \left(1 - \frac{\omega_m}{\omega_c} \right) \sin\phi_c \right]^2, \tag{14}$$

where ω_m and ω_c are the values of $\omega(x)$ in the mantle and core on the respective sides of their border, and ϕ_m and ϕ_c are the phases acquired in the mantle (one layer) and core.

For neutrino trajectories that cross the mantle only ($\phi_c = 0$), Eq. (14) reduces to the adiabatic probability. The passage of neutrinos through the core can lead to an enhancement of the oscillations. As follows from (14), the maximum enhancement can be achieved when $\sin(\phi_c + 2\phi_m)$ and $\sin\phi_c$ are of opposite sign and maximal amplitude: $\sin\phi_c = -\sin(\phi_c + 2\phi_m) = \pm 1$, i.e., when

$$\phi_c = \frac{\pi}{2} + \pi n, \qquad \phi_m = \frac{\pi}{2} + \pi k. \tag{15}$$

Here n and k are integers. In this case the enhancement factor is

$$\frac{P_2^{\rm max}}{\sin^2 2\theta_m} \approx \left(2 - \frac{V_m}{V_c} \right)^2 \approx 2.5, \tag{16}$$

where $\sin^2 2\theta_m$ in the denominator corresponds to the maximum possible transition probability for neutrinos crossing only the mantle, and we have taken into account that $\omega_0 \simeq \omega_m$ and at high energies $\omega_m/\omega_c \approx V_m/V_c$.

The condition (15) and the enhancement described by Eq. (16) are the particular cases of the parametric resonance condition and the parametric enhancement of neutrino oscillations [6,7]. In [13] it was shown that in the case of matter consisting of alternating layers of two different constant densities and (in general) different widths the parametric resonance condition is

$$X_3 \equiv -(\sin\phi_m \cos\phi_c \cos 2\theta_m + \cos\phi_m \sin\phi_c \cos 2\theta_c) = 0, \quad (17)$$

where $\theta_{m,c}$ and $\phi_{m,c}$ are the mixing angles and the acquired oscillations phases in the layers m and c. This condition can also be used as an approximate one when matter density inside the layers is not constant but varies sufficiently slowly. For neutrino oscillations in the Earth we identify the layers m and c with the mantle and core. Since in the energy region $\sin 2\theta\delta \ll V$ one has $\theta_m \approx \theta_c \approx \pi/2$, condition (17) reduces to

$$X_3 \simeq \sin(\phi_m + \phi_c) = 0. \quad (18)$$

Equation (15) is a particular realization of this condition. In the high-energy limit the parametric resonance condition (15) was previously considered in the framework of active-sterile atmospheric neutrino oscillations in [14]. A sizable amplification is also possible if the equality $X_3 = 0$ is realized differently from (15), i.e., when the two terms in (17) do not separately vanish but cancel each other. Interestingly, the parametric resonance condition in Eq. (17) or (18) can indeed be satisfied for neutrino oscillations in the Earth [13–15].

As can be seen in Fig. 1(b), for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations of high-energy neutrinos in the Earth, there are two regions of neutrino zenith angles in which the condition $X_3 = 0$ is satisfied and two prominent peaks appear due to the parametric enhancement: $\cos\Theta \in [-1, -0.93]$ (the inner peak) and $\cos\Theta \in [-0.88, -0.84]$ (the outer peak). The peaks exceed the maximal allowed by the MSW effect value of probability $\sin^2 2\theta_m$ by up to a factor of 2.

For neutrino energies $E \simeq 10$–15 GeV, the oscillation phases corresponding to the inner peak are $\phi_m \simeq \pi/4$, $\phi_c \simeq 7\pi/4$, while for the outer peak they are $\phi_m \simeq 0.35\pi$, $\phi_c \simeq 0.65\pi$. In both peaks to a good accuracy $\phi_m + \phi_c = n\pi$, so that Eq. (18) is satisfied. The phases in the outer peak are closer to the realization (15) of condition (18), and therefore in this peak the parametric enhancement of oscillations is closer to the maximal possible one. From Fig. 1(c) one can see that at $E \simeq 21$ GeV the maximum enhancement condition (15) can be exactly realized in the outer peak. For neutrinos of very high energies ($E \gtrsim 100$ GeV), it can be realized nearly exactly in the inner peak.

Conclusion.—The matter density profile of the Earth is not sufficiently well known. In view of this, our formulas can be used to understand the relevance of various possible features of the profile (e.g., small jumps) for the oscillations of high-energy neutrinos. Namely, they allow us (i) to quantify the effects that uncertainties in the profile will have on the interpretation of the results of future experiments; (ii) to understand what can be learned on the Earth interior from the analyses of atmospheric and accelerator data once the neutrino parameters have been determined; (iii) to study the effects of proper averaging over the energy spectrum of the neutrino beam. In contrast to the oscillation tomography with low energy (e.g., solar) neutrinos, which is sensitive to small structures near the surface of the Earth, high-energy neutrinos can probe large scale structures both in the outer and inner regions of the Earth.

The obtained formulas give a precise and detailed description of the energy and zenith angle dependences of the oscillation effects in the Earth. In particular, we show that the two peaks in the zenith angle distribution of the core-crossing neutrinos (a generic feature at high energies) are due to the parametric enhancement of the oscillations. Observation of these peaks in future high statistics experiments will not only provide an evidence for the parametric resonance, but also (i) confirm the validity of the "standard" theory of oscillations (the standard form of the matter potential at high energies and dynamical features of the oscillations), (ii) give a unique information on the inner regions of the Earth, (iii) provide a cross-check of the values of the oscillation parameters, and (iv) restrict non-standard physics effects.

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