# Maximum Elastic Deformations of Compact Stars with Exotic Equations of State

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I make the first estimates of maximum elastic quadrupole deformations sustainable by alternatives to conventional neutron stars. Solid strange quark stars might sustain maximum ellipticities (dimensionless quadrupoles) up to a few times $10^{-4}$ rather than a few times $10^{-7}$ for conventional neutron stars, and hybrid quark-baryon or meson-condensate stars might sustain up to $10^{-5}$. Most of the difference is due to the shear modulus, which can be up to $10^{33}$ erg/cm$^3$ rather than $10^{30}$ erg/cm$^3$ in the inner crust of a conventional neutron star. Maximum solid strange star ellipticities are comparable to upper limits obtained for several known pulsars in a recent gravitational-wave search by LIGO. Maximum ellipticities of the more robust hybrid model will be detectable by LIGO at initial design sensitivity. A large shear modulus also strengthens the case for starquakes as an explanation for frequent pulsar glitches.

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The LIGO Science Collaboration recently used data from the second LIGO science run (S2) to set upper limits on gravitational-wave emission from 28 known pulsars, 9 of which have no competing upper limit from radio observations [1]. For those 9 pulsars the best S2 upper limits on neutron star ellipticity [a dimensionless quadrupole moment, see Eq. (2)] are a few times $10^{-5}$. With better data now being analyzed, LIGO will soon be sensitive to ellipticities of $10^{-6}$ or less. This raises the question of when a detection might be possible, or when enough nondetections (upper limits) begin to confront some theoretical models of dense matter. The answer depends on the maximum ellipticities sustainable in those models. I make the first estimates of maximum ellipticities for several exotic matter models and find that the LIGO S2 search was already sensitive to the upper end of the theoretical range. LIGO observational results are becoming astrophysically interesting years sooner than previously expected.

The maximum elastic deformation sustainable by a neutron star has been addressed several times in the past few decades—see [2], and references therein. A conventional neutron star consists of a liquid nuclear-matter core covered by a thin solid crust, which is responsible for the deformation and whose microphysics can be extrapolated conservatively from laboratory nuclear physics. More exotic models of compact stars have been proposed, some including large solid cores (see [3] for a summary), but the maximum deformation has not been quantitatively addressed. Historically the problem was of interest first in relation to the "glitch" phenomenon in pulsars, which was believed to be related to starquakes [4]. However, the total elastic energy stored in a maximally strained crust is far too low to explain the strength and frequency of the glitches of the pulsar Vela X-1 [5]. Occasionally works on exotic compact stars have mentioned that solid cores might revive the starquake glitch mechanism, but without estimating numbers.

In this Letter, I estimate maximum elastic deformations sustainable by exotic alternatives to neutron stars and work out the implications for gravitational-wave emission and pulsar glitches. Of the models extant in the literature, solid strange stars allow the largest ellipticities—up to $10^3$ times those of neutron stars—although this model is highly speculative. Hybrid quark-baryon stars and stars with charged meson condensates, both based on more robust theories, might allow ellipticities up to a few times $10^1$ more than those of conventional neutron stars. This makes detectable gravitational-wave emission a prospect for initial LIGO rather than advanced LIGO and makes the starquake model of glitches viable again.

There are several sources of uncertainty in such estimates. The largest is the matter model itself—maximum ellipticities vary by $10^3$ between conventional neutron stars and solid strange stars. The second largest is the breaking strain. I quote fiducial numbers for a breaking strain of $10^{-2}$, which is near the maximum for terrestrial alloys and may be favored by observations of neutron stars in low-mass x-ray binaries, but the breaking strain could be lower by $10^2$–$10^3$ [2]. For hybrid and condensate stars, charge screening might bring the maximum ellipticity down to that of a neutron star. These uncertainties justify making several approximations which simplify the calculations at a cost of introducing relatively small errors as in [2]. Relativistic gravity and rotational effects can change the density profile of a star by tens of percent, but they cancel to some extent and are smaller than the effect of varying the star's mass a few percent [3], and thus I neglect them. Because of the high Fermi energies involved, finite temperature plays a negligible role in determining the maximum ellipticity. In the strange and hybrid stars, a normal solid crust is believed to be still present, but its contribution to the ellipticity is a few percent correction to that of the core. I quote maximum ellipticities including the maximum 200% contribution from the self-gravity of the deformation [2,6], but that could go down by a factor of 2. Further calculations, details, and uncertainties will be presented elsewhere [7].

*Neutron stars.*—Reference [2] computes in its Eq. (69) a maximum $m = 2$ quadrupole moment for a neutron star using a chemically detailed model of the crust. Correcting the definition of shear modulus [6], it reads

$$Q_{22,\max} = 2.4 \times 10^{38}\ \mathrm{g\,cm^2}\left(\frac{\sigma_{\max}}{10^{-2}}\right)\left(\frac{R}{10\ \mathrm{km}}\right)^{6.26} \times \left(\frac{1.4M_\odot}{M}\right)^{1.2}, \tag{1}$$

where $\sigma_{\max}$ is the breaking strain of the crust.

The quadrupole (1) can be converted to the ellipticity $\epsilon = (I_{xx} - I_{yy})/I_{zz}$ used in gravitational-wave papers [1]:

$$\epsilon = \sqrt{8\pi/15}\,Q_{22}/I_{zz}, \tag{2}$$

where the $z$ axis is the rotation axis and $I_{ab}$ is the moment of inertia tensor. For conventional neutron stars, Bejger and Haensel [8] find that the approximation

$$I_{zz} = 9.2 \times 10^{44}\ \mathrm{g\,cm^2}\left(\frac{M}{1.4M_\odot}\right)\left(\frac{R}{10\ \mathrm{km}}\right)^2 \times \left[1 + 0.7\left(\frac{M}{1.4M_\odot}\right)\left(\frac{10\ \mathrm{km}}{R}\right)\right] \tag{3}$$

is accurate to a few percent for a variety of equations of state. Thus we can write the maximum ellipticity of a conventional neutron star as

$$\epsilon_{\max} = 3.4 \times 10^{-7}\left(\frac{\sigma_{\max}}{10^{-2}}\right)\left(\frac{1.4M_\odot}{M}\right)^{2.2}\left(\frac{R}{10\ \mathrm{km}}\right)^{4.26} \times \left[1 + 0.7\left(\frac{M}{1.4M_\odot}\right)\left(\frac{10\ \mathrm{km}}{R}\right)\right]^{-1}. \tag{4}$$

For the fiducial values of mass, radius, and breaking strain, $\epsilon_{\max}$ is $2 \times 10^{-7}$ ($6 \times 10^{-7}$ with self-gravity).

The generalization of Eq. (1) to arbitrary equations of state can be obtained by combining Eqs. (67) and (64) of Ref. [2] as

$$\frac{Q_{22,\max}}{\sigma_{\max}} = \sqrt{\frac{32\pi}{15}}\int dr\,\frac{\mu r^3}{g}\left(48 - 14U + U^2 - \frac{dU}{d\ln r}\right), \tag{5}$$

where $\mu$ is the shear modulus (nonzero only in the solid part of the star), $g$ is the local gravitational acceleration, and $U = 2 + d\ln g/d\ln r$. The two bounding cases are incompressible matter and infinitely compressible matter (a point mass). Note that the latter is equivalent to a conventional neutron star, where the mass of the crust is a small fraction of the mass of the star. For a light crust, $U \ll 1$ and $g \approx GM/r^2$; for the incompressible case, $U = 3$ and $g = GMr/R^3$. If $\mu$ is almost constant [or is replaced by an appropriately averaged value as below Eq. (68) of Ref. [2] ], Eq. (5) simplifies for an incompressible completely solid star to

$$Q_{22,\max} = \gamma\mu R^6\sigma_{\max}/(GM), \tag{6}$$

where $\gamma \approx 13$. Evaluating Eq. (5) for a conventional neutron star with a thin crust and liquid core, $\gamma$ becomes about $120\Delta R/R$, where $\Delta R \approx R/10$ is the thickness of the crust, and thus $\gamma$ is numerically almost identical. The appropriately averaged shear modulus from Ref. [2] is $\mu \approx 4 \times 10^{29}\ \mathrm{erg/cm^3}$, a factor of a few below its maximum value at the bottom of the crust.

*Solid strange stars.*—The idea that some "neutron stars" are in fact made of strange quarks was proposed in the 1970s [9]. The idea that such stars are solid is currently being pursued by Xu's group, beginning with Ref. [10]. (This is distinct from a crystalline color superconducting quark phase [11], which I do not consider here.) Xu notes that the burst oscillation frequencies observed in low-mass x-ray binaries correspond to the first few torsional modes of a solid strange star—if the matter has a typical shear modulus $\mu \approx 4 \times 10^{32}\ \mathrm{erg/cm^3}$, a thousand times the typical value in the crust of a conventional neutron star. Xu estimates that quarks clustered in groups of 18 or so could produce such a shear modulus. Since Ref. [10] was published, the burst oscillation frequency has been observed to closely match the spin frequency of the neutron star in at least one system [12]. This renders the identification with torsional mode frequencies problematic. However, the x-ray burst oscillation mechanism may be different for different binaries, and it is worth considering the effect on the maximum elastic deformation if the shear modulus is very high for whatever reason.

Using Xu's shear modulus in Eq. (6) gives

$$Q_{22,\max} = 2.8 \times 10^{41}\ \mathrm{g\,cm^2}\left(\frac{\mu}{4 \times 10^{32}\ \mathrm{erg/cm^3}}\right)\left(\frac{\sigma_{\max}}{10^{-2}}\right) \times \left(\frac{R}{10\ \mathrm{km}}\right)^6\left(\frac{1.4M_\odot}{M}\right). \tag{7}$$

Bejger and Haensel [8] find a different empirical formula for the moment of inertia for strange stars,

$$I_{zz} = 1.7 \times 10^{45}\ \mathrm{g\,cm^2}\left(\frac{M}{1.4M_\odot}\right)\left(\frac{R}{10\ \mathrm{km}}\right)^2 \times \left[1 + 0.14\left(\frac{M}{1.4M_\odot}\right)\left(\frac{10\ \mathrm{km}}{R}\right)\right]. \tag{8}$$

This combined with Eq. (7) yields a maximum ellipticity

$$\epsilon_{\max} = 2 \times 10^{-4}\left(\frac{\sigma_{\max}}{10^{-2}}\right)\left(\frac{1.4M_\odot}{M}\right)^3\left(\frac{R}{10\ \mathrm{km}}\right)^3 \times \left[1 + 0.14\left(\frac{M}{1.4M_\odot}\right)\left(\frac{10\ \mathrm{km}}{R}\right)\right]^{-1} \tag{9}$$

for solid strange stars, where I have inserted the scalings of $\mu$ from Ref. [10] except for the $f$ and the $x$ dependence, which roughly cancel out. With self-gravity, the canonical number is $\epsilon_{\max} = 6 \times 10^{-4}$.

*Hybrid and meson-condensate stars.*—Glendenning [13] showed that the phase transition from baryonic matter to quark matter occurs over a range of pressures rather than at a single value. (The argument holds for stars with charged meson condensates as well as for stars with quark-baryon cores [3]. The numbers are very similar, so I discuss only hybrid stars.) Purely baryonic matter at high densities is isospin asymmetric, which is energetically unfavorable. Moving toward isospin symmetry (creating more protons) would require negative charges to compensate, and leptons are not favored since they are nearly massless. When the quark phase becomes available, baryonic matter can attain positive charge density by moving negative charge into areas of quark matter.

The crystal structure of the mixed phase changes with density. Immediately above the threshold density for the beginning of the phase transition, the mixed phase consists of small quark droplets arranged in a bcc lattice in a baryonic background. As the density increases, the droplets grow and merge to become rods, then slabs. Eventually the baryonic matter becomes the minority slabs, then rods, then finally droplets before disappearing entirely. The locations of these layers are highly parameter dependent; as an upper limit the mixed-phase crystal can occupy the innermost 8 km of the star [3].

The shear modulus of a bcc lattice of point charges can be written in the parameters of Ref. [3] as [2,5]

$$\mu = 0.075 q^2 D^6 / S^4. \tag{10}$$

I have assumed spherical droplets of (esu) charge density $q$, diameter $D$, and spacing $S$ (Wigner-Seitz cell diameter, or $\sqrt{3}/2$ times the lattice constant). Corrections due to the nonsphericity of the droplets reduce this by an amount that is small in most of the layer. Typical numbers from Chap. 9 of Ref. [3] give

$$\mu = 4 \times 10^{32}\ \mathrm{erg/cm^3} \left(\frac{q}{-0.4e/\mathrm{fm}^3}\right)^2 \left(\frac{D}{15\ \mathrm{fm}}\right)^6 \left(\frac{30\ \mathrm{fm}}{S}\right)^4. \tag{11}$$

This is of order $10^3$ times the typical value in the inner crust, mainly due to the charge of the droplets (about $10^3$ rather than $Z < 55$ in the crust) although the density of droplets is slightly greater too [14].

The dominant correction to Eq. (11) is due to charge screening. This effect is difficult to evaluate precisely, but a rough estimate can be made as follows. Heiselberg, Pethick, and Staubo [15] estimate the screening length

$$\lambda = \left[4\pi \sum_i Q_i^2 \left(\frac{\partial n_i}{\partial \mu_i}\right)\right]^{-1/2} \tag{12}$$

in the mixed phase as 10 fm in the baryonic matter and 5 fm in the quark matter. (Here $Q_i$ is the charge of species $i$, $n_i$ is its number density, and $\mu_i$ is its chemical potential.) Detailed calculations [16] of the partial derivatives in Eq. (12) for baryonic matter without a quark phase suggest that $\lambda \approx 5$ fm is a lower limit. Since these lengths are comparable to the droplet size and separation, screening will appreciably reduce electrostatic effects but not make them negligible. (The leptons can be neglected since their charge density is tiny [3].)

First, note that screening does not appreciably change the droplet size. The quark volume fraction $\chi \approx (D/S)^3$ is set by, e.g., the pressure, and $D$ is found by minimizing the sum of surface and Coulomb energy densities at fixed $\chi$ [3]. The mean charge density $\bar{q}$ is then fixed even under rearrangement of charges due to screening. The Coulomb energy density can be written $g(D/\lambda)C(\chi)D^2$, where $g$ is a geometric factor. Going from uniform density ($D/\lambda \ll 1$) to a shell of charge ($D/\lambda \to \infty$) reduces $g$ by only $1/6$. Using a rough approximation $g \propto 5 + \exp(-D/\lambda)$, the screened $D$ is related to the unscreened $D_0$ by

$$D_0^3 = D^3 + D^4/(10\lambda)\exp(-D/\lambda). \tag{13}$$

Since $D_0$ and $\lambda$ are comparable, this is a few percent correction at most, and since $(D - D_0) \ll \lambda$, even the exponential factor is corrected only by a few percent.

Given that the droplet size does not change appreciably, and given that the charges inside the droplet rearrange themselves respecting spherical symmetry, the problem outside the droplet reduces to the classic screening problem. The potential is multiplied by the Yukawa factor $\exp(-r/\lambda)$. Since the shear modulus is roughly a second derivative of the potential energy, it is multiplied by roughly this factor. For typical $D$ and $S$ values, screening then reduces the shear modulus by $e^3 \approx 20$ ($\lambda \approx 10$ fm) or $e^6 \approx 400$ for $\lambda \approx 5$ fm. The upper limit (weak screening) on the shear modulus is then $\mu \approx 2 \times 10^{31}$ erg/cm$^3$.

The effective shear modulus for the rod and slab configurations can be estimated from the droplet result. Matter made of rods cannot resist a shear stress along the axis of the rods, but will have a perpendicular response similar to that of the droplets. This anisotropic case requires an elastic modulus tensor rather than a shear modulus scalar. However, if the glitch history of the neutron star has led to granulation [3], the formation of small domains with different principal directions, then a macroscopic rms response of the matter averaged over many domains is isotropic with an effective shear modulus reduced by $\sqrt{2/3}$ or $\sqrt{1/3}$, which can be neglected here.

Now evaluate Eq. (5). The density of the core of the star varies only by a factor of a few, so use the incompressible limit. Most of the integral comes from the droplet and rod layers, where the weak-screening shear modulus is roughly constant at $2 \times 10^{31}$ erg/cm$^3$. Then

$$\frac{Q_{22,\max}}{\sigma_{\max}/10^{-2}} = 3.5 \times 10^{39}\ \mathrm{erg/cm^3} \left(\frac{1.4 M_\odot}{M}\right)\left(\frac{R_c}{8\ \mathrm{km}}\right)^6, \tag{14}$$

where $R_c$ is the radius of the hybrid core. Bejger and Haensel [8] find that hybrid stars obey the same moment of inertia relation (3) as normal neutron stars, so

$$\frac{\epsilon_{\max}}{5\times 10^{-6}} = \left(\frac{\sigma_{\max}}{10^{-2}}\right)\left(\frac{1.4M_\odot}{M}\right)^2\left(\frac{R_c}{8\text{ km}}\right)^6\left(\frac{10\text{ km}}{R}\right)^2 \Big/ \left[1+0.7\left(\frac{M}{1.4M_\odot}\right)\left(\frac{R}{10\text{ km}}\right)\right], \tag{15}$$

for a fiducial value of $3\times 10^{-6}$, or up to $9\times 10^{-6}$ with the self-gravity of the deformation.

*Implications.*—What are the immediate consequences for LIGO? The S2 paper [1] quotes direct gravitational-wave observational upper limits on $\epsilon$ for 28 pulsars. However, 19 of these pulsars already have lower indirect upper limits on $\epsilon$ (typically $10^{-8}$ or less) due to the measured spin-downs. The remaining 9 pulsars are in globular clusters where the spin-down is obscured by acceleration, and thus have no competing upper limit. The S2 upper limits on $\epsilon$ for these are 4–24 $\times 10^{-5}$, all within the maximum I estimate for a solid strange star. With LIGO's upcoming data run at full initial sensitivity, the same 9 pulsars will be observable at $\epsilon$ of 1–8 $\times 10^{-6}$, within the maximum for hybrid stars; and the Crab pulsar will be observable at $\epsilon = 1.2\times 10^{-4}$, 6 times less than its spin-down limit and within the solid strange star range [17]. An all-sky search for unknown neutron stars could detect hybrid stars within a kpc and solid strange stars at the galactic core with tens of teraflops computing power [17].

If a pulsar is observed in gravitational waves with $\epsilon \gg 10^{-7}$, it cannot be a conventional neutron star. An upper limit (nondetection) at higher $\epsilon$ does not rule out any exotic model—a given star may happen to be nowhere near its breaking strain. However, with enough strict upper limits, population statistics and deformation mechanisms can be constrained.

Bildsten [18] proposed that the spin frequencies of stars in low-mass x-ray binaries are set by equilibrium between accretion torque and gravitational radiation from thermally induced deformations of the crust. In order to match the observed spin frequencies, this requires quadrupoles $\approx 10^{38}\text{ g cm}^2$. For the exotic stars considered here, such quadrupoles under anisotropic accretion are possible if the breaking strain is smaller than $10^{-3}$ or if the detailed accretion physics (temperature dependence, spreading of material, etc.) prevents achieving breaking strain.

Gravitational waves from freely precessing neutron stars have been considered poor prospects even for advanced LIGO. But if internal damping is weak, a population of stars precessing after birth with $\epsilon = 10^{-4}$ would be detectable with broadband advanced LIGO [19].

A starquake that causes a glitch will also cause a burst of gravitational waves as the modes of the star are excited and ring down. This amplitude is determined by the energy in the glitch, which is determined by the observed frequency jump and thus is not affected by exotic matter models. But the maximum elastic energy in the star scales as the shear modulus, and thus is up to $10^3$ times larger for quark stars than for conventional neutron stars. Vela glitches are still too large and frequent (by several orders of magnitude) to be explained as quakes, but the mean time predicted between quakes is reduced for the Crab pulsar to a few years [5]—comparable to what is observed.

After this Letter was submitted, SGR 1806-20 underwent a giant superflare [20,21] with estimated energy more than $10^{46}$ erg. Theoretical models equate this energy with the maximum elastic energy of the star, which is problematic for a normal crust but feasible with the exotic models considered here.

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