Anderson Localization from the Replica Formalism

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We study Anderson localization in quasi-one-dimensional disordered wires within the framework of the replica σ model. Applying a semiclassical approach (geodesic action plus Gaussian fluctuations) recently introduced within the context of supersymmetry by Lamacraft, Simons, and Zirnbauer, we compute the *exact* density of transmission matrix eigenvalues of superconducting wires (of symmetry class *C*I.) For the unitary class of metallic systems (class *A*) we are able to obtain the density function, save for its large transmission tail.

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At present, there exist two theoretical approaches capable of describing strongly localized phases of disordered wires: supersymmetry (SUSY) [1] and the Dorokhov-Mello-Pereyra-Kumar (DMPK) transfer matrix approach [2]. This represents a serious limitation in as much as both formalisms are ill-suited to generalization to the presence of Coulomb interactions (see, however, Ref. [3]). Reciprocally, it has, so far, not been possible to describe strong localization phenomena by those theories—replica field theory and the Keldish formalism—that may be applied to the analysis of interaction effects. (Exceptions to this rule include successful applications of the replica formalism to certain toy models of localization [4], and the analysis of correlation functions accessible in terms of a *finite* number of replicas [5].) Classical papers such as Refs. [1,6] have, indeed, bolstered the general belief that the replica formalism was inapplicable to nonperturbative localization phenomena for *principal* reasons (related to the notorious zero-replica analytic continuation inherent to the approach.)

It is the purpose of this Letter to show that this view has been overly pessimistic and to introduce a replica field theory approach capable of describing strongly localized phases under rather general conditions. Conceptually, our work is based on a recent paper [7] by Lamacraft, Simons, and Zirnbauer (LSZ) where saddle-point techniques have been applied to analyze the SUSY generating functionals of quasi-one-dimensional disordered conductors. Specifically it was shown that three out of ten symmetry classes of disordered metals are semiclassically exact [8] in that the stationary phase results coincide with those obtained by DMPK methods [11]. We show here that the phenomenon of semiclassical exactness pertains to the replica formalism and, in particular, ''survives'' the analytical continuation inherent to that approach. Applying the technique to the nonsemiclassically exact unitary symmetry class, we find that it still produces qualitatively correct results.

To introduce the replica generalization of the method we consider a disordered superconducting wire in the presence of spin-rotation and time-reversal invariance (symmetry class *C*I in the classification of Ref. [12].) The (thermal) transport properties of this system may be conveniently characterized in terms of the average density of transmission matrix eigenvalues, $\rho(\phi)$. Within the fermion-replica formalism the latter may be expressed through the generating function

$$\mathcal{Z}(\hat{\theta}) \equiv \prod_{a=1}^{R} \det(1 - \sin^2(\theta_a/2)\mathbf{t}\mathbf{t}^\dagger),$$

where $\mathbf{t}\mathbf{t}^\dagger$ is the transmission matrix with eigenvalues $\mathcal{T}_j = \cosh^{-2}(\phi_j/2)$ and $\hat{\theta} \equiv \mathrm{diag}(\theta_1, \ldots, \theta_R)$. Defining the function $F(\theta) \equiv \lim_{R\to 0} \frac{d}{d\theta_1}|_{\theta_a\to\theta} \mathcal{Z}(\hat{\theta})$, the transmission matrix eigenvalue density is obtained as [13] $\rho(\phi) = \frac{1}{2\pi}(F(i\phi + \pi) - F(i\phi - \pi))$.

The field theoretical representation of the generating function for class *C*I is given by

$$\mathcal{Z}(\hat{\theta}) = \int_{g(0)}^{g(T)} \mathcal{D}g e^{-S[g]},$$
$$S[g] = \frac{1}{8}\int_0^T dt\, \mathrm{tr}(\partial g \partial g^{-1}), \tag{1}$$

where g is a field of matrices $g(t) \in \mathrm{Sp}(2R)$, the functional integration extends over the Haar measure on the symplectic group, and $T = L/\xi$ is the length of the wire, L, in units of the localization length ξ. At the left and right end point of the wire the field is subject to boundary conditions [13,14] which in the case of class *C*I read as $g(0) = 1$ and $g(T) = \exp(i\hat{\theta} \otimes \sigma_3)$. Here the Pauli matrix σ_3 acts in the space defining the symplectic condition $g^{-1} = \sigma_2 g^T \sigma_2$.

Our strategy will be to subject the functional (1) to a straightforward stationary phase analysis [15]. Varying the action $S[g]$ with respect to g, one obtains the Euler-Lagrange equation: $\delta_g|_{g=\bar{g}} S[g] = 0 \Rightarrow \partial(\bar{g}^{-1}\partial\bar{g}) = 0$, which integrates to the condition $\bar{g}^{-1}\partial\bar{g} = \mathrm{const}$. The solutions to this latter equation are given by $\bar{g} = \exp(i\bar{W}t/T)$, with constant Lie-algebra elements $\bar{W} \in \mathrm{sp}(2R)$. Evaluating $\bar{g}$ at the system boundary $t = T$, we

obtain the condition $\exp(i\bar{W}) = \exp(i\hat{\theta} \otimes \sigma_3)$. This is solved by $\bar{W} \equiv \hat{\theta}^{(n)} \otimes \sigma_3$, where $\hat{\theta}^{(n)} \equiv \hat{\theta} + 2\pi\hat{n}$ and $\hat{n} = \mathrm{diag}(n_1, \ldots, n_R)$ is a vector of integer "winding numbers." The saddle-point action is given by $S[\bar{g}^{(n)}] = \frac{1}{4T} \times \sum_{a=1}^{R} (\theta_a^{(n)})^2$, indicating that at length scales $T \gtrsim 1$, mean-field configurations traversing multiply around the group manifold become energetically affordable. Physically, these configurations describe the massive (and perturbatively inaccessible) buildup of interfering superconductor diffusion modes. Their proliferation at large length scales forms the basis of the localization phenomenon.

To obtain the contributions of individual saddle points, $\bar{g}^{(n)}$, to the generating function, we need to integrate over quadratic fluctuations. We thus generalize to field configurations $g(t) = \exp(iW(t))\bar{g}^{(n)}$, where the fields $W(t) \in \mathrm{sp}(2R)$ obey vanishing (Dirichlet) boundary conditions $W(0) = W(T) = 0$. Parametrizing these fields as $W = \sum_{\mu=0}^{4} W_\mu \otimes \sigma_\mu$, where $\sigma_0 = 1_2$ and W_μ are $R \times R$ Hermitian matrices subject to the Lie-algebra constraints $W_0 = -W_0^T$ and $W_i = W_i^T$, $i = 1, 2, 3$, the quadratic expansion of the action reads as $S[g] = S[\bar{g}^{(n)}] + S_I[W_0, W_3] + S_{II}[W_1, W_2] + \mathcal{O}(W^3)$, where

$$S_I[W_0, W_3] = \frac{1}{4}\int_0^T dt\,\mathrm{tr}\Big(\partial W_0 \partial W_0 + \partial W_3 \partial W_3 - \frac{2i}{T}(W_0 \partial W_3 + W_3 \partial W_0)\hat{\theta}^{(n)}\Big),$$

$$S_{II}[W_1, W_2] = \frac{1}{2}\int_0^T dt\,\mathrm{tr}\Big(\partial W_1 \partial W_2 + i\frac{\epsilon_{ij3}}{T} W_i \partial W_j \hat{\theta}^{(n)}\Big).$$

The integration over the matrices W_μ leads to fluctuation determinants, which may be calculated by the auxiliary identity $\det(-\partial_t^2 + 2zT^{-1}\partial_t) = \sinh(z)/z$, where $z \in \mathbb{C}$, and the differential operator acts in the space of functions obeying Dirichlet boundary conditions. As a result we obtain the stationary phase generating function

$$\mathcal{Z}(\hat{\theta}) = \sum_{\{n\}} \prod_{a<a'}^{R} \frac{(\theta_a^{(n)} - \theta_{a'}^{(n)})/2}{\sin[(\theta_a^{(n)} - \theta_{a'}^{(n)})/2]} \prod_{a\le a'}^{R} \frac{(\theta_a^{(n)} + \theta_{a'}^{(n)})/2}{\sin[(\theta_a^{(n)} + \theta_{a'}^{(n)})/2]} \times \exp\Big(-\frac{1}{4T}\sum_{a=1}^{R}(\theta_a^{(n)})^2\Big), \quad (2)$$

where the first/second fluctuation factor stems from the integration over the field doublets $(W_0, W_3)/(W_1, W_2)$. [In passing, we note that as an alternative to the brute force integration outlined above the result (2) can be obtained by group theoretical reasoning: according to general principles [10], the fluctuation integral around extremal (geodesic) configurations $\bar{g}^{(n)}$ on a general semisimple Lie group is given by $\prod_{\alpha>0}\alpha(\bar{g}^{(n)})\sin^{-1}(\alpha(\bar{g}^{(n)})) \times \exp(-S[\bar{g}^{(n)}])$, where the product extends over the system of positive roots of the group, $\alpha(\bar{g}^{(n)})$. Equation (2) above is nothing but the $\mathrm{Sp}(2R)$ variant of this formula.]

In the limit of coinciding boundary phases, $\theta_a \to \theta$, the denominators $\sin[(\theta_a^{(n)} - \theta_{a'}^{(n)})/2] \to 0$, i.e., the contribution of configurations $\hat{n}$ containing nonvanishing winding number differences $n_a - n_{a'} \neq 0$ diverges. (At the same time, we do know that the integration over the *full* group manifold must generate a finite result. Indeed, it turns out that if we first sum over all winding number configurations $\hat{n}$ and only then take the limit of coinciding phases, all divergent factors disappear.) This divergence reflects the presence of a zero mode in the system: for uniform boundary phases, $\hat{\theta} \propto 1_R$, transformations $\bar{g}^{(n)} \to \exp(iV^0)\bar{g}^{(n)} \times \exp(-iV^0)$ with constant block-diagonal $V^0 = V_0^0 \otimes \sigma_0 + V_3^0 \otimes \sigma_3$ conform with the boundary conditions but do not alter the action.

As we shall see below, the presence of zero modes implies that only winding number configurations of the special form $(n, 0, \ldots, 0)$ survive the replica limit, $R \to 0$. However, before elaborating on this point, let us evaluate the contribution $\mathcal{Z}_n$ of the distinguished configurations to the generating function. Throughout we will denote the boundary angles by $\theta_a \equiv \theta + \eta_a$, understanding that the limit $\eta_a \to 0$ is to be taken at some stage. (Within this representation, the "free energy" $F(\theta) = \partial_{\theta_1}|_{\theta_a\to\theta}\mathcal{Z}(\hat{\theta}) = \partial_{\eta_1}|_{\eta_a\to 0}\mathcal{Z}(\theta + \hat{\eta})$.) The "dangerous" product $\prod_{a<a'}(\ldots)$ in Eq. (2) then reduces to $\sim(\pi n/\sin(\eta_1/2))^{R-1} \approx (2\pi n/\eta_1)^{R-1}$; all other contributions to $\mathcal{Z}_n$ are finite. The appearance of a pole of $(R-1)$st order hints at the presence of $R-1$ complex zero modes (generated by the $R-1$ components of the matrix V^0 that do not commute with $\hat{g}^{(n)}$). At this stage, we take the limit $R \to 0$. As a result, the divergent factor gets replaced by a "pole of degree (-1)," i.e., the zero: $\eta_1/(2\pi n)$. (It is worth noting that in SUSY a contribution similar to the singularity of degree (-1) is obtained by integration over the noncompact bosonic degrees of freedom; the complementary single replica channel $a = 1$ corresponds to the fermionic sector.) Therefore the subsequent differentiation ($F[\phi] \sim \partial_{\eta_1}|_{\eta_a\to 0}\mathcal{Z}$) *must* act on this linear factor η_1, all other occurrences of η_a in $\mathcal{Z}_n$ may be ignored.

Evaluating the partition function in this manner, we obtain $\mathcal{Z}_{n\neq 0} = \frac{\eta_1}{2\pi n}\frac{\theta+2\pi n}{\theta+\pi n}\exp(-\pi n(\pi n + \theta)/T)$. We finally differentiate with respect to η_1 and arrive at the result $\rho(\phi) = \rho_0(\phi) + \sum_{n\neq 0}\rho_n(\phi)$, where the "Drude plus weak localization term" is $\rho_0 = (2T)^{-1} - (\phi^2 + \pi^2)^{-1}/2$, while the nonperturbative contributions are given by

$$\rho_n(\phi) = -\frac{e^{-\pi^2 n(n+1)/T}}{2\pi^2 n}\,\mathrm{Re}\Big[\frac{\phi + i\pi(2n+1)}{\phi + i\pi(n+1)}\,e^{i(\pi n\phi/T)}\Big]. \quad (3)$$

This expression identically coincides with the SUSY result [7], and with the exact DMPK result [11]. To illustrate the "crystallization" of the transmission matrix eigenvalues at the discrete values $\phi_j \approx 2jT$, the function $\rho(\phi)$ is plotted

in Fig. 1(a) for two values of T. Following LSZ, the heat conductance of the wire may be obtained by integrating the result above against the weight function $1/\cosh^2(\phi/2)$. Summing the result of this integration over winding numbers, one obtains the asymptotic result [7] $g \overset{T\gg 1}{\cong} 4e^{-T}/\sqrt{\pi T}$.

Our analysis so far focused on the specific set of winding number configurations, $(n, 0\ldots, 0)$. To understand why contributions of different structure vanish—a fact that greatly simplifies the formalism—consider the set $(0, \ldots, n, \ldots, 0)$. By symmetry, winding number configurations of this type will lead to an expression similar to $\mathcal{Z}_n$ above, only that the leading prefactor gets replaced: $\eta_1/(2\pi n) \to \eta_a/(2\pi n)$, where $a \in \{2, \ldots, R\}$ marks the position of the nonvanishing winding number. Since, however, we still differentiate with respect to η_1, this contribution vanishes in the limit $\eta_a \to 0$. The argument above may be generalized to generic contributions, $(n_1, n_2, \ldots, n_R) \neq (n, 0, \ldots, 0)$. (By symmetry, one may order the winding numbers in an ascending order $(0, \ldots, 0, 1, \ldots, 1, 2\ldots)$. Assuming that there are N_n winding numbers n (where $\sum_n N_n = R$) and choosing the boundary angle in the sector n to be $\theta + n\eta$, one verifies

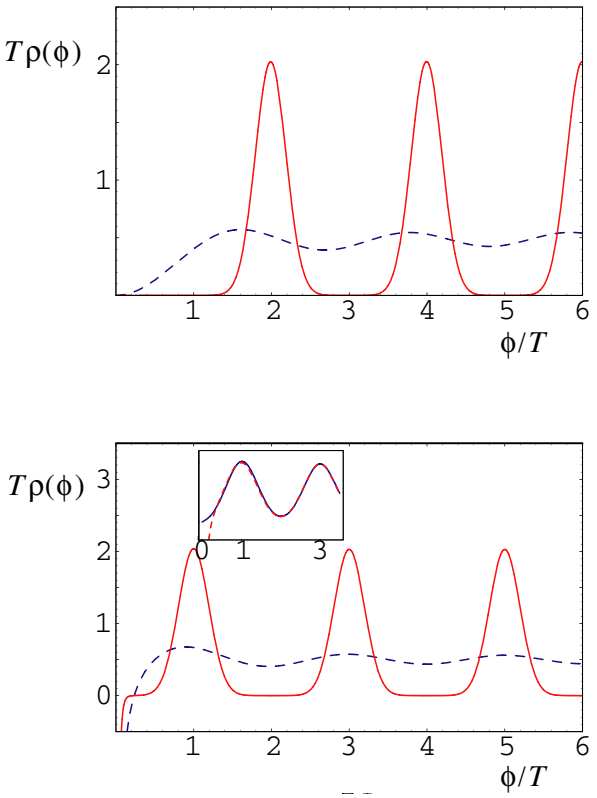

FIG. 1 (color online). Density of transmission eigenvalues of the superconductor class CI (top) and the unitary class A (bottom) for $T = 1$ (dashed lines); and $T = 50$ (solid lines). Inset: the exact [14,21] (solid line) and saddle-point (dashed line) results for class A, $T = 10$. The negative density at small ϕ represents an artifact of the saddle-point approximation.

that for any fixed configuration, the $R \to 0$ result contains uncompensated powers of η and, therefore, vanishes.)

Before proceeding, it is worthwhile to compare the mean-field analysis above to the more established field theory transfer matrix method [1]. To this end, let us interpret $\mathcal{Z}(\hat{\theta}) = \langle g(T)|\exp(-T\hat{H})|1\rangle$ as the path integral describing the (imaginary time) quantum mechanical transition amplitude $|1\rangle \to |g(T)\rangle$ of a particle on the group space $\mathrm{Sp}(2R)$. The Hamiltonian corresponding to the (purely "kinetic") action of the path integral is given by $\hat{H} = -2\Delta$ where Δ is the Laplace operator of the group space $\mathrm{Sp}(2R)$.

Above, we analyzed the transition amplitude in the spirit of a WKB or semiclassical approximation. Alternatively, one may employ the spectral decomposition, $\mathcal{Z}(\hat{\theta}) = \sum_\lambda \psi_\lambda^*(g)\psi_\lambda(1)\exp(-T\epsilon_\lambda)$, where ψ_λ are the eigenfunctions of the Laplace operator, ϵ_λ its discrete energy eigenvalues, and $g \equiv g(T)$. For general Lie groups (and supergroups) formal expressions for these spectral decompositions are known [16]. Noting that for large systems $L \gg \xi$, only eigenstates with minimal energy ϵ_λ effectively contribute to the sum, this knowledge has been used to compute the localization properties of disordered quantum wires within the SUSY formalism [1,14]. The problems with transferring this approach to the replica formalism lie with the analytical continuation from integer group dimension R to $R \to 0$. In taking this limit, it is essential to keep track of high-lying contributions to the spectral sum. These terms grow rapidly more complex which is why attempts to obtain a replica variant of the "quantum approach" above have failed so far.

Having discussed the method for a symmetry class that enjoys the semiclassical exactness, we next outline what happens in cases where this feature is absent. By way of example, consider a metallic disordered quantum wire in the absence of time-reversal invariance—the unitary symmetry class, A. In this case, the fermionic replica generating function is given by $\mathcal{Z}(\hat{\theta}) = \int_{Q(0)}^{Q(T)} \mathcal{D}Q \times \exp(-\frac{1}{8}\int_0^T dt\,\mathrm{tr}(\partial Q)^2)$, where the matrix $Q(t) \in \mathrm{U}(2R)/\mathrm{U}(R)\times\mathrm{U}(R)$ [17,18], and the boundary configurations are given by $Q(0) = \sigma_3 \otimes 1$ and $Q(T) = e^{-i\sigma_2\otimes\hat{\theta}/2}\sigma_3 e^{i\sigma_2\otimes\hat{\theta}/2}$. Here, the two-component structure distinguishes between advanced and retarded indices. As before, the stationary phase configurations: $\bar{Q}(t) = e^{-i\sigma_2\otimes\hat{\theta}^{(n)}t/(2T)}\sigma_3 e^{i\sigma_2\otimes\hat{\theta}^{(n)}t/(2T)}$ of the functional integral do not mix different replica channels. Geometrically, they can be interpreted as trajectories (in general, with nonzero winding number, n) on the meridian of the sphere $\mathrm{U}(2)/\mathrm{U}(1)\times\mathrm{U}(1)$ (the single replica manifold.) Fluctuations may be conveniently parametrized by generalization $\sigma_3 \to e^{iW(t)}\sigma_3$, where $W = W_1 \otimes \sigma_1 + W_2 \otimes \sigma_2$ and $W_{1,2}$ are Hermitian $R \times R$ matrices.

The subsequent calculations largely parallel those for class CI. Expanding to second order in $W_{1,2}$ and

performing the Gaussian integration, we again observe that only winding number configurations $(n, 0, \ldots, 0)$ survive the analytical continuation procedure, $R \to 0$. Differentiating with respect to θ_1 and then putting $\theta_a \to \theta$, we obtain the result $\rho(\phi) = (2T)^{-1} - \sum_{n\neq 0}(-1)^n \rho_n(\phi)$, where

$$\rho_n = \frac{e^{-\pi^2 n(n+1)/T}}{2\pi^2 n} \times \mathrm{Re}\left[\frac{\sqrt{(\phi + i\pi)(\phi + i\pi(2n+1))}}{\phi + i\pi(n+1)} e^{i(\pi n\phi/T)}\right].$$

(The same result is obtained by saddle-point analysis of the SUSY generating functional.) In Fig. 1(b), the function $\rho(\phi)$ is plotted for two values of the parameter T. For large values of T the spectrum crystallizes at $\phi_j \approx (1 + 2j)T$. The lowest eigenvalue ϕ_0 does, indeed, correctly determine the localization length of the system. Except for the evident failure of the method at small values $\phi \ll \phi_0$ [19], the large scale profile of the transmission eigenvalue density is in good agreement with results obtained by the transfer matrix methods [14,20,21].

Summarizing, we have shown how the localization phenomenon in quasi-one-dimensional systems may be described by a semiclassical approach to fermionic-replica field theories. We have been able to reproduce the exact transmission matrix eigenvalue density for symmetry class *C*I, while for the unitary class we obtain qualitatively correct results (except for the tail of the eigenvalue spectrum.) The comparative simplicity of the approach makes us believe that it may be successfully applied to problems that cannot be treated by other means. Evidently, the next direction of research will be the study of the impact of Coulomb interactions on the localization phenomenon.

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