Vortex Lattices in Rotating Atomic Bose Gases with Dipolar Interactions

N. R. Cooper,[1] E. H. Rezayi,[2] and S. H. Simon[3]

[1]TCM Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
[2]Department of Physics, California State University, Los Angeles, California 90032, USA
[3]Lucent Technologies, Bell Laboratories, 600 Mountain View Avenue, Murray Hill, New Jersey 07974, USA
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We show that dipolar interactions have dramatic effects on the ground states of rotating atomic Bose gases in the weak-interaction limit. With increasing dipolar interaction (relative to the net contact interaction), the mean field, or high filling factor, ground state undergoes a series of transitions between vortex lattices of different symmetries: triangular, square, "stripe," and "bubble" phases. We also study the effects of dipolar interactions on the quantum fluids at low filling factors. We show that the incompressible Laughlin state at filling factor $\nu = 1/2$ is replaced by compressible stripe and bubble phases.

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Ultracold atomic Bose gases have emerged as remarkable systems with which to study the unusual response of Bose-condensed systems to rotation. Experiments have allowed the imaging of lattices of quantized vortices [1] and the study of their collective dynamics [2]. New vortex lattice structures in two-component condensates have been observed [3], and a novel regime of vortex density, in which the vortices are closer than the healing length [4], has been accessed [5]. Theory shows that, at very high vortex density [6], atomic Bose gases should undergo a transition into novel uncondensed phases closely related to the incompressible liquids of the fractional quantum Hall effect [4,6–8].

Although Bose condensation can occur for a noninteracting Bose gas, the formation of arrays of quantized vortices under rotation relies on nonvanishing (repulsive) interparticle interactions. In typical atomic Bose condensates, the interactions are so short ranged that they can be viewed as local (contact) interactions. However, significant additional nonlocal interactions can arise if the atoms have intrinsic or induced electric or magnetic dipole moments [9,10]. The recent achievement [11] of the Bose condensation of chromium (which has a large permanent magnetic dipole moment) has opened the door to the experimental study of dipolar-interacting Bose gases. It is important to ask what are the effects of nonlocal interactions on the properties of the vortex lattices.

In this Letter, we study the effects of dipolar interactions on the ground state of a weakly interacting atomic Bose gas under rotation. We show that the additional nonlocal interaction leads to dramatic changes in the nature of the ground state. Within mean-field theory, the triangular vortex lattice predicted for contact interactions is replaced by vortex lattices of different symmetries: square, "stripe," and "bubble" phases. Furthermore, we study the properties at very high vortex density where the ground states for contact interactions are incompressible quantum fluids, characterized by the filling factor [4], $\nu \equiv n/n_V$, where n and n_V are the number densities (per unit area) of particles and vortices. We show that with increasing dipolar interactions the Laughlin state, which is the ground state for pure contact interactions at filling factor $\nu = 1/2$ [4], is replaced by *compressible* states that are well described by stripe and bubble phases.

We consider a system of bosonic atoms of mass M confined to a harmonic trap with cylindrical symmetry about the z axis. We denote the natural frequencies of the trap by $\omega_\parallel$ and $\omega_\perp$ in the axial and transverse directions, and the associated trap lengths by $a_{\parallel,\perp} \equiv \sqrt{\hbar/(M\omega_{\parallel,\perp})}$. The particles are taken to interact through both contact interactions and additional dipolar interactions. We consider the atomic dipole moments to be aligned with the z axis, as in the experiments reported in Ref. [11], such that the net two-body interaction is

$$V(\boldsymbol{r}) = \frac{4\pi\hbar^2 a_s}{M}\delta^3(\boldsymbol{r}) + C_d\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}, \qquad (1)$$

where a_s is the s-wave scattering length arising from contact interactions [12], and C_d is a measure of the strength of the dipolar interactions.

We study the regime of weak interactions, when the mean interaction energy per particle is small compared to the trap energies, $\hbar\omega_{\perp,\parallel}$. The single particle states are then restricted to the ground state of the z confinement (the quasi-2D regime) and to the lowest Landau level (LLL) of the x-y motion [4,13]. While nonrotating gases are typically far from the weak-interaction limit, this limit can be approached at high angular momentum owing to the reduction in particle density through the centrifugal spreading of the cloud [5]. In this limit the interactions are fully specified by the Haldane pseudopotentials [14], V_m: the energy of a pair of atoms in a state with relative angular momentum m, i.e., the expectation value of

$V(\mathbf{r}_1 - \mathbf{r}_2)$ in the state $\phi(\mathbf{r}_1, \mathbf{r}_2) \propto [(x_1 + iy_1) - (x_2 + iy_2)]^m e^{-(x_1^2+y_1^2+x_2^2+y_2^2)/2a_\perp^2 - (z_1^2+z_2^2)/2a_\parallel^2}$. The probability $|\phi|^2$ is maximum at a separation $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{2m}a_\perp$ in the x-y plane, so V_m probes the spatial dependence of $V(\mathbf{r}_1 - \mathbf{r}_2)$. For bosons, symmetry of the wave function means that only even m contribute. The precise form of the dipolar pseudopotentials depends on the trap asymmetry $a_\parallel/a_\perp$. For simplicity of presentation, we study the limit $a_\parallel/a_\perp \to 0$, in which the dipole forces fall off most quickly with increasing separation (increasing m). To leading order in $a_\parallel/a_\perp$, we find

$$V_0 = \sqrt{\frac{2}{\pi}} \frac{\hbar^2 a_s}{M a_\perp^2 a_\parallel} + \sqrt{\frac{2}{\pi}} \frac{C_d}{a_\perp^2 a_\parallel} - \sqrt{\frac{\pi}{2}} \frac{C_d}{a_\perp^3}, \tag{2}$$

$$V_{m>0} = \sqrt{\frac{\pi}{2}} \frac{(2m-3)!!}{m! 2^m} \frac{C_d}{a_\perp^3}. \tag{3}$$

V_0 represents the net *local* interaction, which has contributions from both the contact and dipolar interactions; $V_{m>0}$ represent *nonlocal* interactions which arise only from the dipolar interaction. We consider the relative sizes of the nonlocal and local interactions to be variable, by either tuning the dipolar interaction C_d [9] or tuning a_s close to a Feshbach resonance [15] (which can allow a_s to become negative, letting V_0 be reduced at fixed C_d). We quantify their relative size by the ratio

$$\alpha \equiv \frac{V_2}{V_0} \tag{4}$$

and measure energies in units of V_0 (kinetic energy is frozen out in the 2D LLL). We are interested in bulk properties, so we perform calculations on a periodic rectangular geometry (a torus) which accurately describes the center of an atomic gas where the particle density is approximately uniform. In the weak-interaction limit and for a large number of vortices N_V, the rotation frequency is close to $\omega_\perp$ [5,16] and the number density of vortices is $n_V = 1/(\pi a_\perp^2)$, so a torus with sides a and b contains $N_V = ab/(\pi a_\perp^2)$ vortices.

First, we describe the mean-field ground states. Within Gross-Pitaevskii mean-field theory the ground state is assumed to be fully condensed, with condensate wave function, $\psi(\mathbf{r})$. In the weak-interaction limit, $\psi(\mathbf{r})$ is found by minimizing the mean interaction energy for fixed average particle density, with the wave function constrained to states in the 2D LLL [13]. The problem is mathematically equivalent to the Ginzburg-Landau model for a type-II superconductor close to H_{c2} [17] generalized to a nonlocal interaction. A phenomenological model of this kind has been discussed in the context of superconductivity [18], but we are not aware of general solutions for the vortex lattice ground states. We have found the mean-field ground states for the interactions (1) by numerical minimization on a torus with up to $N_V = 24$ vortices (i.e., exploring periodic states with up to 24 vortices in the unit cell). As a function of α, the ground state undergoes a series of transitions between states of different translational symmetries. Representative images of the particle distributions in these states are shown in Fig. 1 [19].

The ground states we find are as follows: a triangular lattice of single vortices ($0 \leq \alpha \leq 0.20$); a square lattice of single vortices ($0.20 \leq \alpha \leq 0.24$); a stripe phase ($0.24 \leq \alpha \leq 0.60$) [20]. The stripe phase consists of broad lines of high particle density separated by rows of closely spaced vortices. The vortices are ordered along the rows, so the state is a "stripe crystal" with crystalline order in both directions. For $\alpha \geq 0.60$ the states consist of clusters of high particle density arranged in a triangular lattice. Owing to the similarity to crystalline states of electrons in high Landau levels [21], we refer to these as bubble states. We find a sequence of bubble states, which we label by the number q of vortices associated with each bubble. The bubble states we find as ground states are $q = 4$ ($0.60 \leq \alpha \leq 0.91$), $q = 5$ ($0.91 \leq \alpha \leq 1.4$), $q = 6$ ($1.4 \leq \alpha \leq 2.0$), $q = 7$ ($2.0 \leq \alpha \leq 2.7$), and $q = 8$ ($2.7 \leq \alpha$). The particle distributions of the bubble states with $q \geq 4$ resemble Fig. 1(d), with additional vortices confined to the honeycomb network separating the bubbles of particles. We have not looked for states with $q > 8$, but expect that states of arbitrarily large q will appear.

We believe that it should be possible experimentally to access these new vortex lattice ground states. To make contact with experimental parameters, we now examine the case of a spherically symmetric trap ($a_\parallel = a_\perp$). In this case, the triangular lattice is replaced by new ground states

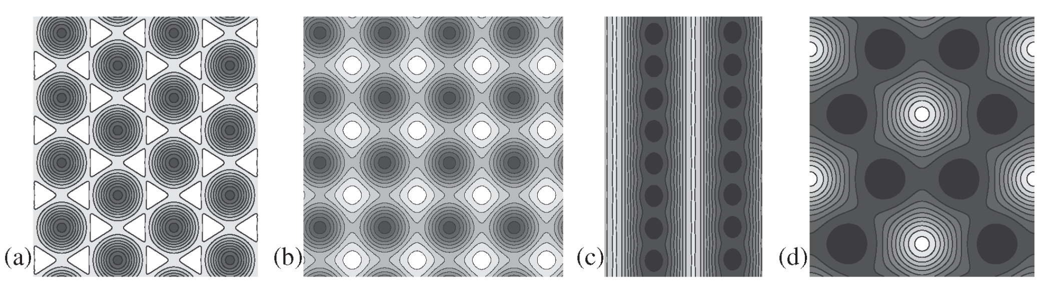

FIG. 1. Contour plots of the particle densities for condensed states on a torus with $N_V = 16$ vortices. The light (dark) shading indicates high (low) particle density (on arbitrary scales). (a) Triangular vortex lattice ($a/b = \sqrt{3}/2$); (b) square vortex lattice ($a/b = 1.0$); (c) stripe phase (shown for $\alpha = 0.528$, $a/b = 0.608$); (d) $q = 4$ bubble phase ($a/b = \sqrt{3}/2$).

if a_s is tuned [15] to values $a_s \lesssim -0.11 C_d M/\hbar^2$. Note that, despite this negative value of a_s, the dipolar interaction makes V_0 positive, which, in the weak-interaction limit, is sufficient to ensure stability to collapse. Under these circumstances, for a mean particle density n_{3d}, the interaction energy per particle is of order $C_d n_{3d}$, so the weak-interaction limit requires $n_{3d} \lesssim \hbar\omega_\perp / C_d$ [16]. For the value of C_d for chromium [11], which has a magnetic dipole moment of 6 Bohr magnetons, and taking $\omega_\perp = 2\pi \times 100\ \mathrm{rad\,s^{-1}}$, the weak-interaction limit should be a good approximation for densities less than about $10^{14}\ \mathrm{cm}^{-3}$.

We now turn to discuss the ground states beyond mean-field theory. In Ref. [6] it was shown that the parameter controlling the validity of mean-field theory for a rotating atomic Bose gas is the filling factor, $\nu \equiv n/n_V$. For contact interactions, the triangular vortex lattice predicted by mean-field theory was shown to be destroyed by quantum fluctuations for $\nu < \nu_c^{\mathrm{tri}}$, with $\nu_c^{\mathrm{tri}} \sim 6$ [6], and replaced by new ground states which include incompressible liquids closely related to fractional quantum Hall states [4,6–8].

We have studied the effects of dipolar interactions on these strongly correlated ground states using exact diagonalization studies. We now return to the case of $a_\parallel / a_\perp \to 0$ and focus our attention on the filling factor $\nu = 1/2$, for which the exact ground state for contact interactions ($\alpha = 0$) is the $\nu = 1/2$ bosonic Laughlin state [4]. This is the dominant incompressible state of rotating bosons with contact interactions. We find that the exact ground state in the presence of dipolar interactions is well described by the Laughlin state up to $\alpha \simeq 0.5$. At this point there is an abrupt transition in the ground state and its overlap with the Laughlin state falls to a very small value [Fig. 2(a)]. Our studies show that the states that replace the Laughlin state at $\alpha \gtrsim 0.5$ are compressible phases which are well described by the mean-field ground states discussed above. Here we present evidence of a stripe phase at $\alpha = 0.528$ and of the $q = 4$ bubble phase at $\alpha = 0.758$. (For larger α, we find evidence of bubble phases with larger q.) To identify these broken-symmetry states we make extensive use of the classification of energy eigenstates by a conserved momentum[22]. We express the momentum as a dimensionless vector $\boldsymbol{K} = (K_x, K_y)$ using units of $2\pi\hbar/a$ and $2\pi\hbar/b$ for the x and y components, and report only positive K_x, K_y [states at $(\pm K_x, \pm K_y)$ are degenerate by symmetry]. We identify the broken translational symmetry of the ground states by making use of the fact that this leads to the appearance of quasidegeneracies in the spectrum at wave vectors equal to the reciprocal lattice vectors of the crystalline order [23].

Evidence of stripe ordering at $\alpha = 0.528$ is presented in Fig. 2(b). This shows the excitation spectrum on a torus with an aspect ratio chosen to be consistent with the mean-field ground state. The quasidegeneracy of the ground state at $\boldsymbol{K} = (0, 0)$ with a state at $(0, 3)$ indicates a strong tendency to translational symmetry breaking in the stripe pattern found in mean-field theory: three stripes lying parallel to the short axis of the torus. However, unlike the mean-field state, we find no evidence of crystalline order parallel to the stripes: the ground state appears to be a "smectic" [24] in which translational symmetry is broken in only one direction.

Evidence for the formation of the $q = 4$ bubble phase is shown in Fig. 2(c). At filling factor $\nu = 1/2$, each bubble contains $\nu q = 2$ particles, so this state can also be described as a triangular lattice of pairs of particles. The low energy states shown in Fig. 2(c) as solid circles correspond to wave vectors at the reciprocal lattice vectors of the $q = 4$ bubble state [shown in Fig. 1(d)]. Although these states do not seem to be cleanly separated from the rest of the spectrum, the other low energy states (square symbols) can be understood as arising from a nearby (in energy) configuration in which the crystal is rotated by 90°.

The appearance of stripe and bubble states at $\nu = 1/2$ indicates that these states are much more stable to quantum fluctuations than is the triangular vortex lattice (which is unstable for $\nu < \nu_c^{\mathrm{tri}} \sim 6$ [6]). This enhanced stability can be understood within a simple Lindemann analysis, in which one asserts that quantum melting occurs when the quantum fluctuations of a lattice site exceed a multiple c_L of the lattice spacing. Treating the fluctuations of each vortex independently, one expects the triangular vortex lattice to melt for $\nu < \nu_c^{\mathrm{tri}} = \sqrt{3}/(2\pi c_L^2)$ [6,25]. For the

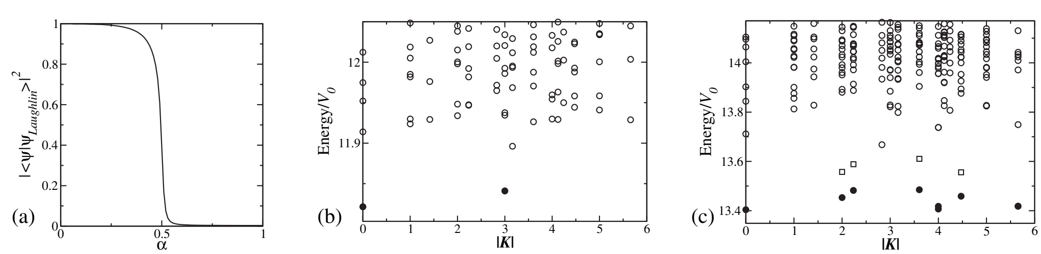

FIG. 2. Results of exact diagonalization studies at $\nu = 1/2$. (a) Overlap of the exact ground state with the Laughlin state ($N = 8$, $N_V = 16$, torus aspect ratio $a/b = 1$). (b) Energy spectrum for $\alpha = 0.528$ ($N = 9$, $N_V = 18$, $a/b = 0.82$). The solid circles are for momenta $\boldsymbol{K} = (0, 0)$ and $(0, 3)$, showing evidence for a stripe state. (c) Energy spectrum at $\alpha = 0.758$ ($N = 8$, $N_V = 16$, $a/b = \sqrt{3}/2$). The solid circles are at $\boldsymbol{K} = (0, 0)$, $(4, 0)$, $(0, 4)$, $(4, 4)$, $(2, 0)$, $(1, 2)$, $(3, 2)$, $(2, 4)$, which are the reciprocal lattice vectors of the $q = 4$ bubble state. The square symbols are at $\boldsymbol{K} = (0, 2)$, $(2, 1)$, $(2, 3)$, $(4, 2)$, which are reciprocal lattice vectors of a rotated crystal.

square lattice, we find $\nu_c^{\rm sq} = 1/(\pi c_L^2) = (2/\sqrt{3})\nu_c^{\rm tri}$, close to that for the triangular lattice. The enhanced stabilities of the stripe and bubble phases arise from the existence of larger length scales and larger numbers of particles per unit cell (as compared to the triangular or square vortex lattices). For the q bubble phase, the bubbles form a triangular lattice with lattice constant $\sqrt{q(2\pi/\sqrt{3})}a_\perp$. Applying the Lindemann analysis to the quantum fluctuations of the center of mass of a bubble, we find $\nu_c^q = (\sqrt{3}/2\pi c_L^2) \times (1/q^2) = \nu_c^{\rm tri}/q^2$. Even for the smallest ($q = 4$) bubble state, this critical filling factor is very much smaller than that for the triangular lattice. For the stripe phases, there are two length scales: the intervortex separation in the directions parallel, $R_\parallel$, and perpendicular, $R_\perp$, to the stripes. One therefore expects two transitions: when fluctuations of the vortices along the stripes exceed $c_L R_\parallel$, there is a loss of order in that direction, leading to a smectic phase; when fluctuations perpendicular to the stripes exceed $c_L R_\perp$, the stripe ordering will finally be lost. Assuming that, for this final transition, of order $R_\perp/R_\parallel$ vortices must fluctuate together, we find that loss of stripe order should occur at $\nu_{c,\perp}^{\rm stripe} = (1/\pi c_L^2)(R_\parallel/R_\perp)^2 = (2/\sqrt{3})(R_\parallel/R_\perp)^2\nu_c^{\rm tri}$. Over the range $\alpha = 0.24$–0.60 for which the stripe is the mean-field ground state the ratio $R_\perp/R_\parallel = 1$–2.59, so the critical filling factor for the stripe can be as small as about $(1/6)\nu_c^{\rm tri}$. While it would be desirable to have a more complete Lindemann analysis in which the collective modes [26] of these new lattices are quantized, we believe that the simple analyses presented here capture the essential physics of the relative stability of the states to quantum fluctuations.

At filling factors above $\nu = 1/2$, quantum fluctuations of the stripes and bubbles are strongly suppressed. For the cases in Figs. 2(b) and 2(c), increasing the number of particles to $N = 12$ (so that the respective filling factors are $\nu = 2/3$ and $3/4$) leads to much improved ground state quasidegeneracies. Quantum fluctuations are enhanced for $\nu < 1/2$. For small nonzero α, we find incompressible liquids at filling factors, $\nu = p/(3p \pm 1)$, expected for composite fermions [8] formed from bosons bound to *three* vortices. The state at $\nu = 1/4$ is well described by the $\nu = 1/4$ Laughlin state. For $\alpha \gtrsim 1.7$ this state is replaced by bubble phases, consistent with the expectation from the Lindemann analysis that for large q these can have critical filling factors less than $1/4$.

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Note added.—Following the online publication of this work, a study of the same mean-field regime appeared [27]. The results of the revised version of [27] are now consistent with our predictions for the square and stripe phases.

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