Particle Scattering in Loop Quantum Gravity

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We devise a technique for defining and computing n-point functions in the context of a background-independent gravitational quantum field theory. We construct a tentative implementation of this technique in a perturbatively finite model defined using spin foam techniques in the context of loop quantum gravity.

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The lack of a general technique for computing particle scattering amplitudes is a seriously missing ingredient in nonperturbative quantum gravity [1,2]. Various problems can be traced to this absence: the difficulty of deriving the low energy limit of a theory, of comparing alternative theories, such as alternative versions of the Hamiltonian operator in loop quantum gravity (LQG) or different spin foam models, or comparing the predictions of a theory with those of perturbative approaches to quantum gravity, such as perturbative string theory. Here we explore one possibility for defining a general formalism aimed at computing scattering amplitudes. We outline a calculation strategy, which can certainly be improved. Our interest is not in a particular theory, but rather in a general technology to be used for analyzing different models. For concreteness, we implement this strategy in the context of a specific model, and present a well-defined and perturbatively finite expression, which, under substantial assumptions and approximations, might be interpreted as a general-covariant n-point function.

In conventional QFT, we can derive all scattering amplitudes from the n-point functions

$$W(x_1, \ldots, x_n) = Z^{-1} \int D\phi\, \phi(x_1) \ldots \phi(x_n) e^{-iS[\phi]}, \quad (1)$$

where the x_i, $i = 1, \ldots, n$ are points of the background spacetime, ϕ is the quantum field, $S[\phi]$ its action, and Z is the integral of the sole exponential of the action. Alternatively, the n-point functions can be derived from their Euclidean continuations, defined without the i factor above. The integral (1) is well defined in perturbation theory or as a limit of a lattice regularization, under appropriate renormalization. A well-known difficulty of background-independent quantum field theory is that if we assume (1) to be well defined with general-covariant measure and action, then the n-point function is easily shown to be constant in spacetime [see *f.i.* [3]]. This is the difficulty we address here.

Consider a spacetime region R such that the points x_i lie on its 3D boundary Σ. Call φ the restriction of the field ϕ to Σ. Then (1) can be written in the form

$$W(x_1, \ldots, x_n) = Z^{-1} \int D\varphi\, \varphi(x_1) \ldots \varphi(x_n) W[\varphi, \bar{\Sigma}] W[\varphi, \Sigma], \quad (2)$$

where (in the Euclidean case)

$$W[\varphi, \Sigma] = \int_{\phi|_\Sigma = \varphi} D\phi_R e^{-S_R[\phi_R]} \quad (3)$$

is the integral over the fields in R bounded by the 3D field φ on Σ, and S_R is the restriction of the action to R; while $W[\varphi, \bar{\Sigma}]$ is the analogous quantity defined on the complement $\bar{R}$ of R in spacetime. The field boundary propagator (3) has been considered in [1,4] and studied in [5]. Assume that we are interested in an amplitude (1) of an interacting theory approximated by a free (Gaussian) theory $S^{(0)}[\phi]$ in some regime, and that the interaction term in the action can be restricted to R. For instance, we are interested in the scattering of some particles and the scattering region is inside R. Then we can replace $W[\varphi, \bar{R}]$ with its free theory equivalent

$$W_0[\varphi, \bar{\Sigma}] = \int_{\phi|_\Sigma = \varphi} D\phi_{\bar{R}} e^{-S^{(0)}_{\bar{R}}[\phi]} \equiv \Psi_\Sigma[\varphi]. \quad (4)$$

This integral is Gaussian and can be performed, giving a Gaussian "boundary state" $\Psi_\Sigma[\varphi]$, determined by appropriate boundary conditions for the field at infinity. For instance, if we take R to be defined by $t > 0$, then $\Psi_{t=0}[\varphi]$ is the conventional vacuum state in the functional Schrödinger representation. In general, we expect the boundary state to be given by some Gaussian functional of the boundary field φ on Σ.

Consider now a diffeomorphism invariant theory including the gravitational field. Assume that the equations above hold, in some appropriate sense. The field ϕ represents the gravitational field, as well as any eventual matter field, and we assume action and measure to be diffeomorphism invariant. Two important facts follow [6]. First, because of diffeomorphism invariance the boundary propagator $W[\varphi, \Sigma]$ is independent from (local deformations of) the surface Σ. Thus in gravity the left-hand side of (3) reads $W[\varphi]$. Second, the geometry of the boundary surface Σ is not determined by a background geometry (there is not any), but rather by the boundary gravitational field φ itself.

We can obtain an indication on the possible forms of the boundary state in gravity from the free quantum theory of noninteracting gravitons on Minkowski space. If we take R to be $t > 0$, for instance, then $\Psi_{t=0}[\varphi]$ must be approximated by the well-known Schrödinger vacuum wave functional of linearized gravity. This is a Gaussian state picked around a classical geometry: the flat geometry of the $t = 0$ surface in Minkowski space. In the case of a compact R, it is then reasonable to consider a Gaussian boundary state $\Psi_q[\varphi]$ picked around *some* 3-geometry q of the boundary surface Σ. Thus, we may expect an expression of the form

$$W(x_1, \ldots, x_n; q) = Z^{-1} \int D\varphi \varphi(x_1) \ldots \varphi(x_n) \Psi_q[\varphi] W[\varphi] \tag{5}$$

to approximate (1) when the interaction term can be neglected outside R. For this equation to be significant, we have to fix the meaning of the coordinates x_i, since the rest of the expression is generally covariant. There is an obvious choice: the points x_i can be defined with respect to the geometry q. For instance, if $n = 4$, $t_1^0 = t_2^0 = 0$, and $t_3^0 = t_4^0 = T$ [we use $x = (t, \vec{x})$] we can take q to be the geometry of a rectangular box of height T and side L and interpret $\vec{x}_i$ as proper distances from the boundaries of the box. In other words, the localization of the arguments of the n-point function can be defined with respect to geometry over which the boundary state is picked. Notice that x_i in (5) are then *metric* coordinates: they refer to gravitational field values. They are not anymore general-covariant coordinates as in (1). In this manner, we can give meaning to n-point functions in a background-independent context.

Physically, we can interpret R as a finite spacetime region where a scattering experiment is performed. The quantities x_i are then relative distances and relative proper time separations, measured along the boundary of this region, and determined by (the mean value of) the gravitational field (hence the geometry) on this boundary. This is the general-relativistic definition of position measurements in a realistic scattering experiment [see [1]].

In order to give (5) a fully well-defined meaning, and compute n-point functions concretely, we need four ingredients: (i) a proper definition of the space of the 3D fields φ integrated over, and a well-posed definition of the integration measure. (ii) An explicit expression for the boundary propagator $W[\varphi]$. (iii) An explicit expression for the boundary state $\Psi_q[\varphi]$. (iv) A definition of the field operator $\varphi(x)$. In the following, we analyze the status of these four ingredients in the loop and spin foam approach to quantum gravity. We consider for simplicity pure gravity without matter.

(i) In quantum theory, the boundary values of Feynman integrals can be taken to be the classical dynamical variables only if the corresponding operators have continuum spectrum. If the spectrum is discrete, the boundary values are the quantum numbers that label a basis of eigenstates [see [1]]. In our case, the boundary field φ represents the metric of a 3D surface. Let us assume here the LQG results that the 3D metric is quantized. Therefore we must replace the continuum gravitational field variable φ with the quantum numbers labeling a basis that diagonalizes some metric degrees of freedom. These can be taken to be (abstract) spin networks s, or s knots. An s knot is here an equivalence class under [extended [7]] diffeomorphisms of embedded spin networks S. An embedded spin network is a graph immersed in space, labeled with spins and intertwiners [see [1]]. The s knots are discrete [7]. Thus, we rewrite (5) in the form

$$W(x_1, \ldots x_n; g) = Z^{-1} \sum_s c(s) \varphi_s(x_1) \ldots \varphi_s(x_n) \Psi_q[s] W[s], \tag{6}$$

where the meaning of $\varphi_s(x)$ will be specified later on. The discrete measure $c(s)$ on the space of the s knots is defined by the projection of the scalar product of the space of the embedded spin networks [$c(s) = 1$, except for discrete symmetries of s]. For simplicity, and in order to match with the spin foam formalism that we use below, we restrict here the space of the spin networks to the four-valent ones and we identify spin networks with the same graph, spins, and intertwiners (i.e., we ignore knotting and linking).

(ii) The boundary propagator $W[s]$ is a now a function of a boundary spin network. A possibility is to identify it with the boundary propagator $W[s]$ defined in the spin foam formalism [8]; that is, to identify the spin foam boundary states s with the s-knot states of LQG [9]. For concreteness, let us choose here the model defined by the $SO(4)/SO(3)$ group field theory [10], which gives a perturbation expansion finite at all orders [11]. This is the model denoted GFT/C in [1]. [A problem with this choice is that the GFT/C spins label representations of the self-dual and anti-self-dual $SO(3)$ subgroups of the $SO(4) = SO(3) \times SO(3)$ decomposition; while in the well-defined version of LQG, which is based on a real connection, they label representations of the spatial rotations $SO(3)$ subgroup of $SO(4)$. There is a version of LQG based on the self-dual gauge group—in fact, the original one—but it is still very poorly understood. Here, however, we disregard these issues, since our aim is not to single out the physically correct theory, but only to show that a background-independent definition of n-point functions is available.] The amplitude of a spin network s is given in this GFT/C by

$$W[s] = \int D\Phi f_s[\Phi] e^{-\int \Phi^2 - \lambda \int \Phi^5}. \tag{7}$$

Here Φ is a function on $[SO(4)]^4$ and the precise meaning of the (symbolic) integrals in the exponent is detailed in [1,8]. The quantity $f_s[\Phi]$ is a polynomial in the field Φ, determined by s. It is defined by picking one factor

$$\Phi^i_{\alpha_1\ldots\alpha_4} = \int dg_1 \ldots dg_4 \Phi(g_1, \ldots, g_4) R^{(j_1)\beta_1}_{\alpha_1} \times (g_1) \ldots R^{(j_4)\beta_4}_{\alpha_4}(g_4) v^i_{\beta_1\ldots\beta_4} \quad (8)$$

per node of s, where v^i is the intertwiner of the node and $j_1, \ldots, j_4$ the colors of the adjacent links, and contracting the indices α_i according to the connectivity of the graph of s. The expression (7) is well defined and finite order by order in λ. [The rigorous proof of this statement is complete up to certain degenerate graphs [8].] The explicit computation of $W[s]$ is entirely combinatorial and can be performed in terms of combinations of nJ Wigner symbols [1]. For completeness, recall that the reason for the definition (7) is that the expansion of $W[s]$ in λ can be written as a sum over spin foams bounded by the spin network s

$$W[s] = \sum_{\partial\sigma=s} A(\sigma), \quad (9)$$

where the spin foam amplitude $A(\sigma)$ is the Barrett-Crane discretization of the exponential of the Einstein-Hilbert action of the discrete four geometry defined by the spin foam σ. Therefore the definition (7) of $W[s]$ can be interpreted as a (background-independent) discretization of the functional integral (3).

(iii) An expression for the boundary state $\Psi_q[s]$ can be obtained from the analysis of the coherent states in LQG [12–15]. For concreteness, let us pick here Conrady's definition of a coherent state [15]. Other more refined expression could be used instead. Conrady has defined a state $\Psi_0[S]$ that describes the Minkowski vacuum as a function of embedded spin networks S, under certain approximations and assumptions. This function has the property of being picked on spin networks that are "weaves," namely, that approximate a flat metric q when averaged over regions large compared to the Planck scale [16]. This vacuum state can be written as follows. Pick Cartesian coordinates $x^a, a = 1, 2, 3$, on a 3D surface equipped with a flat metric q and with total volume V. Fix a triangulation $\mathcal{T}$ of lattice spacing a, small compared to the Planck length l_p in the metric q. Restrict the attention to embedded spin networks S living on $\mathcal{T}$. Define the form factor of a spin network as

$$F^{ab}_S(\vec{x}) = \frac{\pi l_P^4}{96a^3} \sum_{v\in S} \sum_{e\in v} \int_0^1 dt \int_0^1 dt' \dot{e}^a(t) \dot{e}^b(t') \delta(\vec{x} - \vec{x}_v), \quad (10)$$

where v are the vertices of the spin network S, $\vec{x}_v$ their position, $e{:}t \mapsto e^a(t)$ the edges, $e \in v$ indicates the edges e adjacent to the vertex v, and $\dot{e}^a = de^a/dt$. Its Fourier transform is $F^{ab}_S(\vec{k}) = V^{-1/2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} F^{ab}_S(\vec{x})$. Then

$$\Psi_0[S] = \mathcal{N} \exp\bigg[-\frac{1}{4l_p^2} \sum_{\vec{k}} |\vec{k}| |F^{ab}_S(\vec{k}) j_e(j_e + 1) - \sqrt{V} \delta^{ab} \delta_{\vec{k},0}|^2 \bigg], \quad (11)$$

where the momenta summed over are the discrete modes on the triangulation, j_e is the spin associated to the edge e, and $\mathcal{N}$ is a normalization factor. To understand this construction, notice that if we consider the gravitational field associated to the spin network (in the sense of the weaves), $q^{ab}_S(\vec{x}) = F^{ab}_S(\vec{x}) j_e(j_e + 1)$, then $\Psi_0[S] = \Psi_0[q_S]$ where

$$\Psi_0[q] = e^{-1/4\hbar\kappa \int d^3x \int d^3y [(q^{ab}(\vec{x}) - \delta^{ab}) W_\Lambda(\vec{x}-\vec{y})(q^{ab}(\vec{y}) - \delta^{ab})]} \quad (12)$$

is the Schödinger functional representation of the linearized vacuum state. $W_\Lambda(\vec{x} - \vec{y})$ is a lattice regularization of the vacuum covariance. We can extend this construction to a 3D (Euclidean) rectangular boundary Σ simply taking the product of the Conrady states associated to each of the eight faces forming Σ.

We need to carry this result over to the diffeomorphism invariant s-knot states. Given an abstract spin network s there will be in general one embedded spin network $S(s)$ that maximizes the state $\Psi_0[S]$. We can then tentatively define $\Psi_0[s] = \Psi_0[S(s)]$. Notice that if $\Psi_0[S]$ is picked on weaves, then the diffeomorphism invariant state $\Psi_0[s]$ defined is picked on the corresponding ("weavy") discrete 3 geometries. The maximization condition can be interpreted as a gauge choice, picking the coordinate system in which the 3 geometry is closest to the Euclidean metric. The gauge invariant state is then chosen to be the restriction of the state to this gauge surface. In the spirit of [15], we restrict to embedded spin networks S living on $\mathcal{T}$. Given an s-knot s, there is only a discrete number of such spin networks that are in the class s: we choose $S(s)$ that maximizes (11) among these. We expect this definition (possibly with an appropriate correction of the Conrady vacuum state) to converge for fine triangulations, making the background structure chosen effectively irrelevant for a triangulation sufficiently finer than the Planck length. This construction provides a finite definition of $\Psi_0[s]$, diffeomorphism invariant by definition.

An alternative root for defining the vacuum functional $\Psi_0[s]$ might be obtained from an appropriate $T \to \infty$ limit of $W[s]$, following [6,17].

(iv) Finally, we need to define the field $\phi(x)$ in (6). Following [15] we write $h^{ab}_s(x) = (q^{ab}_{S(s)}(x) - \delta^{ab})$, where the point x is defined in terms of the boundary metric q and $q^{ab}_{S(s)}$ is defined above (12). An alternative, which we do not pursue here, is to derive $h^{ab}(x)(S)$ from the action of two $SU(2)$ generators [18].

We can now combine the various pieces discussed. Take a 3D metric space (Σ, q) isometric to the boundary of a parallelepiped in 4D Euclidean space with height T and cubic base of side L. Fix a triangulation of Σ. The simplest choice is to start from a cubic triangulation of Σ, and obtain a four-valent lattice, by splitting each (six-valent) vertex of the cubic lattice into two vertices. Replacing the various items discussed into the formal expression (5) we obtain

$$W^{a_1b_1\ldots a_nb_n}(x_1,\ldots x_n; L, T) = Z_{LT}^{-1}\sum_s c(s) h_s^{a_1b_1}(x_1)\ldots h_s^{a_nb_n}(x_n)\Psi_q[s]W[s], \quad (13)$$

where the sum is over all the s knots that can be embedded in the triangulation. The normalization factor is the "vacuum to vacuum" amplitude $Z_{LT} = \sum_s c(s)\Psi_q[s]W[s]$. Equation (13) can be expanded in powers λ^n. n is the number of vertices of the spin foam, which is the number of 4 simplices in a simplicial complex dual to the spin foam, if this exists. As a rough estimate, we can imagine each 4 simplex to have Planck size: if classical configurations dominate, the main contribution should come from n of the order of the 4 volume of the interaction region in Planck units.

All quantities in (13) are well defined. The expression is likely to be finite at any order in λ. We can take (13) as a tentative definition of an n-point function within the formalism of nonperturbative quantum gravity. [On "particle" states on finite spacial regions, see [19].] More precisely, we can consider (13) as a definition of the formal expression

$$W^{a_1b_1\ldots a_nb_n}(x_1,\ldots,x_n) = Z^{-1}\int Dg\, g^{a_1b_1}(x_1)\ldots g^{a_nb_n}(x_n)e^{-S_{EH}[g]}, \quad (14)$$

where S_{EH} is the Einstein-Hilbert action, computed at relative spacetime distances $x_1,\ldots,x_n$ evaluated in terms of the mean value of the quantum gravitational field itself, on a box encircling the interaction region. The construction considered opens a natural possibility for the Euclidean continuation: to take $T \to iT$, but we know no solid justification for this at present.

Hypotheses and simplifications used to get to (13) are severe. Many issues remain open, in particular: the identification of the spin networks of LQG with the spin foam ones, the consistency between the finiteness of the interaction region and the long range properties of gravity, the analytic continuation and the proper definition of the vacuum state. Our aim here, therefore, is far less ambitious than to determine the physically correct theory. It is to show that a background-independent definition of n-point functions can be given and may lead to perturbatively finite expressions in a model. The open problem is thus to determine the consistent and physical correct set of quantities $W(s)$, $\Psi_q[s]$ and $h_s^{ab}(x)$ entering (13). The question is then whether (13) is indeed finite, convergent, and independent from the auxiliary structures used to define it, when the triangulation is sufficiently finer than the Planck scale, and whether this construction leads, in a first approximation, to the general relativity scattering tree amplitudes.

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