High-Density Limit and Inflation of Matter

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We prove rigorously that for a nonvanishing probability of having electrons in matter, with Coulomb interactions, within a sphere of radius R, the latter, *necessarily*, grows not any slower than $N^{1/3}$ for large N, where N denotes the number of electrons. Thus it is not surprising that matter occupies so large a volume.

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Undoubtedly, one of the most important and serious problems that quantum physics has resolved is that of the stability of matter. The Pauli exclusion principle turns out to be not only sufficient for stability but also necessary. Early investigations of the stability of matter go to the classic work of Dyson and Lenard [1] and more recently to the monumental work of Lieb and Thirring [2,3]. Systematic studies of matter without the exclusion principle, referred to as "bosonic matter," have been also carried out [2,4–9] over the years. The drastic difference between matter (with the exclusion principle) and "bosonic matter," with Coulomb interactions, is that the ground-state energy E_N for the former $-E_N \sim N$, with N denoting the number of negative charges (the electrons), while for the latter $-E_N \sim N^\alpha$, where $\alpha > 1$. Such a power law behavior with $\alpha > 1$ implies that instability as the formation of a single system consisting of $(2N + 2N)$ particles is favored over two separate systems brought together, each consisting of $(N + N)$ particles, and the energy released upon the collapse of the two systems into one, being proportional to $[(2N)^\alpha - 2N^\alpha]$, will be overwhelmingly large for realistic large N, e.g., $N \sim 10^{23}$. In regard to matter without the exclusion principle, it is interesting to quote Dyson [4]: "[Bosonic] matter in bulk would collapse into a condensed high-density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb.... Matter without the exclusion principle is unstable." It is important to note that the collapse of "bosonic matter" is not [8] a characteristic of the dimensionality of space and that such matter does not change, for example, from an "implosive" to a "stable" or to an "explosive" phase with the change of dimensionality. In regard to the exclusion principle, or more generally to the spin and statistics connection, it is interesting to also quote from the translator's Preface [10] of the classic book by Tomonaga on spin: "The existence of spin, and the statistics associated with it, is the most subtle and ingenious design of Nature—without it the whole universe would collapse."

The purpose of this communication is to establish the following key result concerning the stability of matter. We prove rigorously that for a nonvanishing probability of having the electrons in matter within a sphere of radius R, the latter, *necessarily*, grows not any slower than $N^{1/3}$ for large N. Here it is worth recalling the words addressed by Ehrenfest to Pauli in 1931 on the occasion of the Lorentz medal [cf. [4]] to this effect: "We take a piece of metal, or a stone. When we think about it, we are astonished that this quantity of matter should occupy so large a volume." He went on by stating that the Pauli exclusion principle is the reason: "Answer: only the Pauli principle, no two electrons in the same state."

The Hamiltonian under consideration is taken to be the N-electron one

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j}^{N} \frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} - \sum_{i=1}^{N}\sum_{j=1}^{k} \frac{Z_j e^2}{|\mathbf{x}_i - \mathbf{R}_j|} + \sum_{i<j}^{k} \frac{Z_i Z_j e^2}{|\mathbf{R}_i - \mathbf{R}_j|}, \quad (1)$$

where m denotes the mass of the electron, and $\mathbf{x}_i$, $\mathbf{R}_j$ correspond, respectively, to positions of electrons and nuclei. Also we consider neutral matter $\sum_{i=1}^{k} Z_i = N$.

We first derive an upper bound to the expectation value of the kinetic energy of the electrons. Let $|\Psi(m)\rangle$ denote a normalized state giving a strictly negative expectation value for the Hamiltonian, i.e.,

$$-\mathcal{E}_N[m] \leq \langle\Psi(m)|H|\Psi(m)\rangle < 0, \quad (2)$$

where $-\mathcal{E}_N[m] = E_N < 0$ is the ground-state energy [11], and we have emphasized its dependence on the mass m of the electron. Here we note, in general, that a part of a negative spectrum does not necessarily coincide with bound states. By definition of the ground-state energy, the state $|\Psi(m/2)\rangle$ cannot lead for $\langle\Psi(m/2)|H|\Psi(m/2)\rangle$ a numerical value lower than $-\mathcal{E}_N[m]$. That is,

$$-\mathcal{E}_N[m] \leq \langle\Psi(m/2)|H|\Psi(m/2)\rangle. \quad (3)$$

Accordingly, if we denote the interaction part in (1) by V, we have

$$-\mathcal{E}_N[2m] \leq \left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V \right| \Psi(m) \right\rangle. \quad (4)$$

Upon writing

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + \left(\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V \right), \qquad (5)$$

the extreme right-hand side of the inequality (2) then leads to

$$\left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} \right| \Psi(m) \right\rangle < -\left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V \right| \Psi(m) \right\rangle \qquad (6)$$

which from (3) gives

$$T \equiv \left\langle \Psi(m) \left| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \right| \Psi(m) \right\rangle < 2\mathcal{E}_N[2m]. \qquad (7)$$

This inequality may be equivalently obtained in the following manner. If we define the energy functional $\langle \phi | H | \phi \rangle = E_\phi$ for any normalized state $|\phi\rangle$ such that $E_\phi < 0$, then by using (5) we may write $E_\phi = (1/2)T_\phi + E'_\phi$, where $E'_\phi = \langle \phi | H' | \phi \rangle$ and H' denotes the second term in (5) within the brackets defining a Hamiltonian with mass $2m$. This immediately gives $T_\phi < 2|E'_\phi|$ and from the definition of the ground-state energy $-\mathcal{E}_N[2m]$ as the infimum of the spectrum in a theory with the mass m replaced by $2m$ [p. 36 of Ref. [3]] leads to the inequality in (7).

The explicit lower bound for the ground-state energy of matter [2] with the mass of the electron multiplied by 2, together with the Lieb-Thirring lower bound for the kinetic energy [2], then lead from (7) to the basic inequalities [12]:

$$\frac{9}{5} \frac{\hbar^2}{m} \int d^3(\mathbf{x}) \rho^{5/3}(\mathbf{x}) < T < 5\left(\frac{me^4}{\hbar^2}\right) N[1 + Z^{2/3}]^2, \qquad (8)$$

where $Z|e|$ corresponds to the nucleus carrying the largest positive charge, $\rho(\mathbf{x})$ in (8) is the particle density

$$\rho(\mathbf{x}) = N \sum_{\sigma_1,\ldots,\sigma_N} \int d^3\mathbf{x}_2 \ldots d^3\mathbf{x}_N |\Psi(\mathbf{x}\sigma_1, \mathbf{x}_2\sigma_2, \ldots, \mathbf{x}_N\sigma_N)|^2, \qquad (9)$$

and $\int d^3\mathbf{x} \rho(\mathbf{x}) = N$, with a sum in (9) over spin indices σ_i. Since there are no elements of arbitrary high atomic number Z, the latter is always bounded. This is related to stability, as it implies [9] that Z remains bounded for $N \to \infty$. That is, as more and more matter is put together, thus increasing the number N of electrons, the number k of nuclei in such matter, as separate clusters, would necessarily increase and not arbitrarily fuse together and their individual charges remain bounded, i.e., technically, as $N \to \infty$, then stability implies that $k \to \infty$ as well, and no nucleus may be found that would carry an arbitrary large portion of the total positive charge available. Let $\mathbf{x}$ denote the position of an electron relative, for example, to the center of mass of the nuclei. Let $\chi_R(\mathbf{x}) = 1$, if $\mathbf{x}$ lies within a sphere of radius R, and $= 0$ otherwise. Then clearly for the probability of the electrons to lie within such a sphere we have

$$\begin{aligned} \text{Prob}\,[|\mathbf{x}_1| \le R, \ldots, |\mathbf{x}_N| \le R] &\le \text{Prob}\,[|\mathbf{x}_1| \le R] \\ &= \frac{1}{N} \int d^3\mathbf{x} \chi_R(\mathbf{x}) \rho(\mathbf{x}) \\ &\le \frac{1}{N} \left[\int d^3\mathbf{x} \rho^{5/3}(\mathbf{x}) \right]^{3/5} (v_R)^{2/5}, \end{aligned} \qquad (10)$$

where in the last inequality, we have used Hölder's inequality, the fact that $[\chi_R(\mathbf{x})]^{5/2} = \chi_R(\mathbf{x})$, and where $v_R = 4\pi R^3/3$.

From (8) and (10), we have the explicit inequality

$$\text{Prob}\,[|\mathbf{x}_1| \le R, \ldots, |\mathbf{x}_N| \le R] \left(\frac{N}{v_R} \right)^{2/5} < \left(\frac{1}{a_0^3} \right)^{2/5} 1.846[1 + Z^{2/3}]^{6/5}, \qquad (11)$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius, and is the main result of the Letter. We may infer from (11) the inescapable fact that necessarily for a nonvanishing probability of having the electrons within a sphere of radius R, the corresponding volume v_R grows not any slower than the first power of N for $N \to \infty$, since otherwise the left-hand side of (11) would go to infinity and would be in contradiction with the finite upper bound on its right-hand side. That is, *necessarily*, the radius R grows not any slower than $N^{1/3}$ for $N \to \infty$, establishing the result stated above. Thus it is not surprising that matter occupies so large a volume. In turn, the infinite density limit $N/v_R \to \infty$ does not arise as the probability on the left-hand side of (11) would go to zero in this limit. [We note in passing that the above analysis gives the following lower bound for the *expectation* value

$$\left\langle \sum_{i=1}^{N} \frac{|\mathbf{x}_i|}{N} \right\rangle > 0.1052 a_0 \frac{N^{1/3}}{[1 + Z^{2/3}]} \qquad (12)$$

consistent with earlier well-known estimates (cf. p. 36 of Ref. [3])].

The method developed above has been also used to analyze the localizability and stability of other quantum mechanical systems [13].

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[11] That the negative spectrum of H is not empty easily follows by noting that $-\mathcal{E}_N[m]$ is bounded above by $-(me^4/2\hbar^2)\sum_{i=1}^{k} Z_i^2$.

[12] Here we have used an improved constant value in the Lieb-Thirring inequality as obtained by D. Hundertmark, A. Laptev, and T. Weidl, Inventiones Mathematicae **140**, 693 (2000). Within the original Lieb-Thirring inequality [2], a wider range for T is obtained with the constants $9/5$, 5 in (8) replaced, respectively, by $(3/10)(3\pi/4)^{2/3}$, 16.6208. Also for the original inequality, the constants 1.846, 0.1052 in (11) and (12), respectively, may be replaced by 10, 0.02575. Needless to say, the conclusions reached in this Letter hold for either set of constants.

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