How to Detect the Fourth-Order Cumulant of Electrical Noise

Joachim Ankerhold[1,2] and Hermann Grabert[1]
[1]*Physikalisches Institut, Albert-Ludwigs-Universität, 79104 Freiburg, Germany*
[2]*Service de Physique de l'Etat Condensé, DSM/DRECAM, CEA Saclay, 91191 Gif-sur-Yvette, France*
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It is proposed to measure the current noise generated in a mesoscopic conductor by macroscopic quantum tunneling (MQT) in a current biased Josephson junction placed in parallel to the conductor. The theoretical description of this setup takes into account the complete dynamics of detector and noise source. Explicit results are given for the specific case of current fluctuations in an oxide layer tunnel junction, and it is shown how the device allows to extract the fourth-order cumulant of the noise from the MQT data for realistic experimental parameters.

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Within the last decade electrical noise has moved into the focus of research activities on electronic transport in nanostructures [1], since it provides information on microscopic mechanisms of the transport not available from the voltage dependence of the average current. Lately, attention has turned from the noise autocorrelation function to higher order cumulants of the current fluctuations characterizing non-Gaussian statistics [2,3]. While theoretical attempts to predict these cumulants for a variety of devices are quite numerous [3], experimental observation is hard because of small signals, large bandwidth detection, and strict filtering demands. A first pioneering measurement by Reulet *et al.* [4] of the third cumulant of the current noise from a tunnel junction has intensified efforts and several new proposals for experimental setups have been put forward very recently, some of which are based on Josephson junctions (JJ) as noise detectors. Lindell *et al.* [5] employed a Coulomb blockaded JJ to demonstrate that the conductance of the junction in the Coulomb gap region is sensitive to the non-Gaussian character of noise applied to the junction. A modification of this setup was suggested by Heikkilä *et al.* [6] to get specific information on the third cumulant of the noise. Another recent experiment [7] has observed activated-over-the-barrier jumps of a JJ biased by a noisy current. The data are consistent with resonant activation produced by the second cumulant of the noise at the plasma frequency of the junction. For a measurement of the full distribution of current fluctuations Tobiska and Nazarov [8] suggested to use an array of overdamped JJs acting as a threshold detector for rare current fluctuations triggering over-the-barrier jumps. Since in the overdamped limit retrapping spoils the buildup of a detectable voltage, it was argued by Pekola [9] that an experimentally more accessible detector would extract the noise characteristics from modifications of the macroscopic quantum tunneling (MQT) rate in an underdamped JJ. A consistent theory, however, is still elusive.

In all experimental setups to measure higher order cumulants realized and proposed so far, heating is one of the major experimental obstacles [10]. Thus, experiments have primarily attempted to establish just the unspecified non-Gaussian nature of the noise or to measure the third cumulant (skewness). This one is particularly accessible since it can be discriminated from purely Gaussian noise due to its asymmetry, e.g., when inverting the current through the conductor. In contrast, the fourth-order cumulant (sharpness) on the one hand due to heating effects may be completely hidden behind the second and the third one, but on the other hand is required to gain an essentially complete characterization of the distribution of current fluctuations. In this Letter we propose and analyze a setup, with the circuit diagram depicted in Fig. 1, which allows us to detect the fourth-order cumulant of the current noise generated by a nanoscale conductor. Since this conductor is placed in parallel to a current biased JJ in the zero voltage state, no heating occurs prior to the decay of this state by MQT. However, the MQT rate is modified in a specific way by the even higher order cumulants characterizing the non-Gaussian current fluctuations of the conductor.

The complete statistics of current noise generated by a mesoscopic conductor can be gained from the generating functional

$$G[\phi] = e^{-S_G[\phi]} = \left\langle \mathcal{T} \exp\left[\frac{i}{e}\int_C dt I(t)\phi(t)\right]\right\rangle,$$

where $I(t)$ is the current operator and $\mathcal{T}$ the time ordering operator along the Kadanoff-Baym contour C. Time

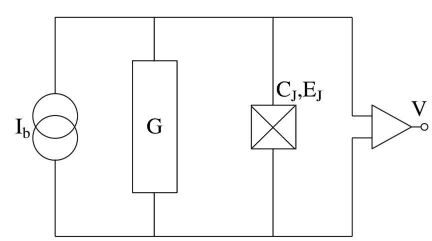

FIG. 1. Electrical circuit containing a mesoscopic conductor G in parallel to a JJ with capacitance C_J and coupling energy E_J biased by an external current I_b. The switching out of the zero voltage state of the JJ by MQT is detected as a voltage pulse V.

correlation functions of arbitrary order of the current are determined from functional derivatives of $G[\phi]$; in particular, the average current

$$C_1(t) = \langle I(t)\rangle = ie\partial S_G[\phi]/\partial\phi(t)|_{\phi=0}$$

and the current autocorrelation function

$$C_2(t,t') = \langle I(t)I(t')\rangle = e^2\partial^2 S_G[\phi]/\partial\phi(t)\partial\phi(t')|_{\phi=0}.$$

Higher order functional derivatives give the cumulants related to non-Gaussian current fluctuations

$$C_n(t_1,\ldots,t_n) = -(-ie)^n\partial^n S_G[\phi]/\partial\phi(t_1)\cdots\partial\phi(t_n)|_{\phi=0}.$$

We remark that the functional $S_G[\phi]$ carries the full frequency dependence of all current cumulants and not just their time averaged zero frequency values usually studied in the field of full counting statistics [2].

By way of example let us consider an Ohmic resistor of resistance R in thermal equilibrium at inverse temperature β. Then, the functional $S_G[\phi] \equiv S_R[\phi]$ reads

$$S_R[\phi] = 2\rho\int_C dt\int_C dt'\alpha(t-t')\phi(t)\phi(t') \tag{1}$$

where $\rho = h/(4\pi e^2 R)$ and

$$\alpha(t) = \frac{\pi}{2(\hbar\beta)^2\sinh^2(\pi t/\hbar\beta)}. \tag{2}$$

The quadratic form reflects the Gaussian nature of the current fluctuations in this case which implies that all cumulants except for C_2 vanish. On the other hand, for a tunnel junction with many transmission channels, where each channel has a small transmission coefficient T_i leading to the dimensionless conductance $g_T = h/(4\pi e^2 R_T) = \pi\sum_i T_i$, where R_T is the tunneling resistance, one has [11]

$$S_T[\phi] = -4g_T\int_C dt\int_C dt'\alpha(t-t')\sin^2\left[\frac{\phi(t)-\phi(t')}{2}\right]. \tag{3}$$

Here, the periodicity in ϕ reflects the discreteness of the transferred charges associated with non-Gaussian current fluctuations.

To gain information on the noise of the conductor, it may be placed in parallel to a current biased JJ as depicted in the circuit diagram of Fig. 1. For a bias current I_b below the critical current I_c, the JJ is in its zero voltage state and the bias current flows as a supercurrent entirely through the JJ branch of the circuit. Consequently, no heating occurs in the conductor and the total system can easily be kept at low temperatures, where the decay of the zero voltage state occurs through MQT. The rate of this process depends with exponential sensitivity on the current fluctuations of the conductor so that the JJ acts as a noise detector.

The MQT rate Γ can be calculated in the standard way [12,13] from the imaginary part of the free energy F, i.e., $\Gamma = (2/\hbar)\,\mathrm{Im}\{F\}$, where $F = -(1/\beta)\ln(Z)$ is related to the partition function $Z = \mathrm{Tr}\{e^{-\beta H}\}$. In the path integral representation one has

$$Z = \int\mathcal{D}[\theta]e^{-S[\theta]},$$

which is a sum over all imaginary time paths with period $\hbar\beta$ of the phase difference θ across the JJ weighted by the dimensionless action $S[\theta] = S_{JJ}[\theta] + S_G[\theta/2]$. Here

$$S_{JJ}[\theta] = \frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\left[\frac{1}{2}\varphi_r^2 C_J\dot\theta(\tau)^2 + U(\theta)\right] \tag{4}$$

is the action of the bare JJ and S_G is the generating functional introduced above. In Eq. (4) $\varphi_r = \hbar/2e$ denotes the reduced flux quantum, C_J is the capacitance of the JJ, and the tilted washboard potential $U(\theta) = -E_J[\cos(\theta) - s\theta]$, where E_J is the Josephson energy and $s = I_b/I_c$. The factor of 2 in the argument of S_G arises from the fact that the voltage across the conductor equals the voltage $V_J = (\hbar/e)(\dot\theta/2)$ across the JJ.

In the MQT regime the partition function of the isolated JJ is dominated by the so-called bounce trajectory, an extremal $\delta S[\theta] = 0$ periodic path in the inverted barrier potential. By approximating a well-barrier segment of $U(\theta)$ around a well minimum θ_m by a harmonic + cubic potential, $V(\delta\theta) = (M\Omega^2/2)\delta\theta^2(1 - \delta\theta/\delta\theta_0)$ with $\delta\theta = \theta - \theta_m$, one finds an analytic solution in the limit of vanishing temperature, i.e., $\theta_B(\tau) = \delta\theta_0/\cosh^2(\Omega\tau/2)$. Here, Ω is the frequency for small oscillations around the well bottom, $M = \varphi_r^2 C_J$, and $\delta\theta_0$ denotes the exit point determined from $U(\theta_m) = U(\theta_m + \delta\theta_0)$. The corresponding MQT rate reads

$$\Gamma_0 = 6\sqrt{6\Omega V_b/\hbar\pi}\exp\left(-\frac{36}{5}\frac{V_b}{\hbar\Omega}\right)$$

where $V_b = (2M\Omega^2/27)\delta\theta_0^2$ is the barrier height.

Following the theory of the effect of an electromagnetic environment on MQT [12,14], the partition function can now be calculated for arbitrary coupling between detector and conductor based on a numerical scheme developed in Ref. [13]. Analytical progress is made when the noise generating element has a dimensionless conductance $g_T \ll E_J/\hbar\Omega$ so that the influence of the noise can be calculated by expanding about the unperturbed bounce which gives

$$\Gamma = \Gamma_0 e^{-S_G[\theta_B/2]}. \tag{5}$$

The correction $S_G[\theta_B/2]$ is usually dominated by the second cumulant C_2 (width) and the fourth cumulant C_4 (sharpness). Note that this treatment still contains the full dynamics of detector and noise source since any approximation relying on a time scale separation, such as, e.g., the adiabatic limit considered in Ref. [9], is usually not applicable.

Now, in case of a tunnel junction [15] as noise element one finds for $S_G[\theta_B/2] \equiv S_T[\theta_B/2]$ from (2) and (3)

$$S_T[\theta_B/2] = \frac{g_T}{4\pi}\int_0^\infty d\omega\,\omega|\tilde{\rho}(\omega)|^2 \tag{6}$$

with

$$\tilde{\rho}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\theta_B(\tau)/2} e^{i\omega\tau}.$$

By expanding the first exponential and performing the Fourier transform for each power of $\theta_B(\tau)$ separately, the relevant part $\rho(\omega) = \tilde{\rho}(\omega) - 2\pi\delta(\omega)$ reads

$$\rho(\omega) = \frac{\pi}{4}\frac{\omega}{\sinh(\pi\omega/\Omega)}\sum_{k=1}^{\infty}\frac{(2i\delta\theta_0)^k}{k!(2k-1)!}\prod_{l=1}^{k-1}\left(\frac{\omega^2}{\Omega^2}+l^2\right). \tag{7}$$

This way, Eq. (6) can be cast into

$$S_T\left[\frac{\theta_B}{2}\right] = \frac{g_T}{4\pi^3}\sum_{k,k'=1}^{\infty}\frac{(-1)^{(3k+k')/2}2^{k+k'}A_{kk'}}{k!k'!(2k-1)!(2k'-1)!}\delta\theta_0^{k+k'} \tag{8}$$

with the coefficients

$$A_{kk'} = \int_0^\infty dy\frac{y^3e^y}{(e^y-1)^2}\left[\prod_{l=1}^{k-1}\left(\frac{y^2}{4\pi^2}+l^2\right)\right]\left[\prod_{l=1}^{k'-1}\left(\frac{y^2}{4\pi^2}+l^2\right)\right].$$

Here, $A_{kk'} = A_{k'k}$ so that in (8) only terms contribute with $k + k'$ even. This means that all odd cumulants of the fluctuating current vanish according to a vanishing net current $\langle I(t)\rangle = 0$ through the conductor. Specifically, one finds

$$A_{11} = 6\zeta(3), \qquad A_{22} = 6\zeta(3) + \frac{5!\zeta(5)}{2\pi^2} + \frac{7!\zeta(7)}{16\pi^4},$$
$$A_{31} = 24\zeta(3) + 5\frac{5!\zeta(5)}{4\pi^2} + \frac{7!\zeta(7)}{16\pi^4}.$$

The terms in the sum (8) related to a contribution of order $\delta\theta_0^{k+k'}$ determine the impact of the $(k+k')$th moment of the current fluctuations of the tunnel junction onto the MQT process. Since in Eq. (7) the term of order $\delta\theta_0^k$ contains contributions centered around $\omega \approx 0, \Omega \ldots, k\Omega$, the influence of the $(k+k')$th moment results from mode mixing between fluctuations with frequencies $l\Omega$ and $l'\Omega$ where $l \leq k, l' \leq k'$.

In lowest order, $k + k' = 2$, one gains from Eq. (8) the Gaussian noise contribution providing a correction to the bare MQT rate

$$\Gamma_T^{(2)} = \Gamma_0 \exp\left[-\frac{6\zeta(3)g_T}{\pi^3}\delta\theta_0(s)^2\right]. \tag{9}$$

Apparently, this reflects the well-known fact that Gaussian noise leads to a *reduction* of the tunneling rate [12].

At order $\delta\theta_0^4$ the sum (8) gives three contributions, namely, $k = 1$, $k' = 3$, and $k = 3$, $k' = 1$ with $A_{13} = A_{31}$, as well as $k = 2$, $k' = 2$ with A_{22}. This leads to

$$\Gamma_T^{(4)} = \Gamma_T^{(2)} \exp\left[\frac{4g_T}{\pi^3}(2A_{31} - A_{22})\delta\theta_0(s)^4\right], \tag{10}$$

so that the fourth-order cumulant of the current noise contains both, fluctuations that suppress tunneling (related to A_{22}) and fluctuations that increase MQT (related to A_{31}). Since $2A_{31} - A_{22} > 0$, the total impact of the fourth moment leads to an *enhancement* of the MQT rate.

To obtain explicit results, the harmonic + cubic potential $V(\delta\theta)$ leads for the amplitude of the bounce to $\delta\theta_0(s) = 3\sqrt{1-s^2}/s$. Accordingly, the barrier height scales with the dimensionless current s as $V_b(s) = (2E_J/3)(1-s^2)^{3/2}/s^2$, and the plasma frequency reads $\Omega(s) = [\sqrt{2E_JE_C}/\hbar](1-s^2)^{1/4}$ with charging energy $E_C = 2e^2/C$. Experimentally, in the standard procedure [16] to measure the MQT rate, a bias current pulse of height I_b and duration t is adiabatically turned on and a voltage pulse is detected when the JJ switches to its finite voltage state. This procedure is performed a few thousand times to built-up switching histograms that determine the switching probabilities

$$P(s) = 1 - e^{-\Gamma(s)t}.$$

In Fig. 2 these so-called s curves are shown for various values of the duration of the current pulse t. The parameters chosen are accessible in realistic experiments. Apparently, for short pulses when the switching occurs for values of the bias current close to the critical current of the JJ, the effect of the non-Gaussian noise fluctuations is completely suppressed due to a decreasing amplitude $\delta\theta_0(s)$ of the bounce. However, for longer pulses (or equivalently, for shorter pulses in the tails towards lower s values) the fourth-order cumulant leads to a substantial influence, mostly dominated by a shift to smaller s values compared to an Ohmic resistor with identical Gaussian noise contribution. As seen from Eq. (10) this is due to

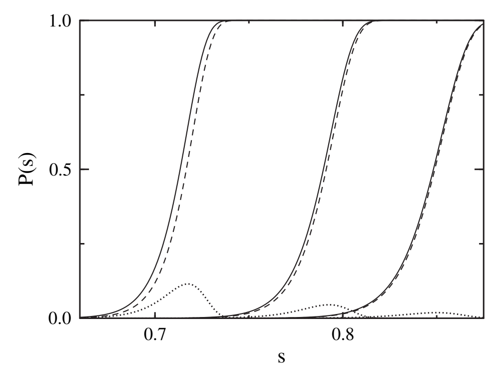

FIG. 2. Switching probabilities out of the zero voltage state ("s curve") of a JJ in parallel to a tunnel junction (solid lines) and to an ohmic resistor (dashed lines) with identical second cumulant for various pulse lengths of the bias current, from left to right: 1 ms, 1 μs, 10 ns. Dotted lines display the differences of the corresponding switching probabilities. Parameters are $\sqrt{E_J/E_C} = 10$, $g_T = 2$, $\Omega(s=0) = 100$ GHz.

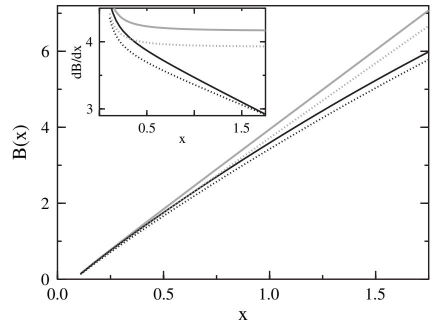

FIG. 3. $B(x) = -\ln[\Gamma(x)/\Gamma_0(x)]$ vs $x = (1 - s^2)/s^2$ (dimensionless bias current s) for a tunnel junction (black) and an ohmic resistor (gray) with identical second cumulant. Solid lines display the situation in the absence, dashed lines in the presence, of additional Gaussian noise in the wiring with $R/R_T = 0.05$. The inset displays the corresponding slopes $dB(x)/dx$. Parameters are the same as in Fig. 2.

an effective decrease of the barrier height by the non-Gaussian fluctuations of the tunnel junction. The contrast in the switching probabilities between MQT in the presence of purely Gaussian noise and in the presence of a tunnel junction is larger than 10% for values of the pulse height around $s = 0.73$, a consequence of the exponential sensitivity of the JJ in the MQT range even to weak non-Gaussian fluctuations.

For the on-chip detection circuit proposed here, the impact of the fourth-order cumulant on the s curves needs to be clearly discriminated from effects of purely Gaussian noise. This is achieved by considering the function

$$B(x) = -\ln[\Gamma(x)/\Gamma_0(x)]$$

with the variable $x = (1 - s^2)/s^2$ which allows us to discriminate between weak Gaussian and non-Gaussian noise due to a qualitatively different scaling behavior with varying x. Note that also in the standard analysis of escape rates a scaling property is exploited to determine the junction parameters E_J and Ω (and thus the bare rate Γ_0) from the s dependence of V_b at high temperatures [16]. With this information at hand, the exponential dependence of the MQT rate on the environment can be used to probe the noise in the circuit at low temperatures. Figure 3 illustrates that for $B(x)$ purely Gaussian noise results essentially in a straight line, while non-Gaussian noise displays a nonlinear behavior. Even more pronounced are the differences in the slopes $dB(x)/dx$, which saturate for larger x values when only Gaussian noise is present, but strongly decrease with increasing x in the presence of a nonlinear conductor. Hence, the derivative $dB(x)/dx$ shows directly the impact of higher than second order cumulants in the noise fluctuations. Most importantly, this scaling property is robust, since it holds for any sort of Gaussian or non-Gaussian noise. Namely, additional Gaussian noise present in the wiring, incorporated by an additional resistor with resistance $R \ll R_T$ [cf. Equation (1)], merely shifts $dB(x)/dx$ and thus does *not* spoil the scaling behavior originating from C_4 (see Fig. 3). Finally, by fitting $dB(x)/dx$ with Eq. (9), even the coefficient $2A_{31} - A_{22}$ related to C_4 in Eq. (10) can be extracted.

The formulation developed above is completely general and applies to *any* nanoscale conductor in parallel to a JJ. A systematic expansion of the action $S_G[\theta_B/2]$ in powers of $(\theta_B/2)^n$ determines the dynamical impact of the nth order cumulant C_n onto the MQT process. To summarize, we have proposed a nanoelectrical circuit where a JJ placed in parallel to an arbitrary conductor acts as detector for non-Gaussian current noise. Since no net current flows through the noise source, heating effects are suppressed and one obtains access to the even order cumulants of the distribution function which are notoriously difficult to detect. For experimentally realistic parameters we have explicitly shown how the fourth-order cumulant of a tunnel junction can be extracted.

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