Calculation of the One- and Two-Loop Lamb Shift for Arbitrary Excited Hydrogenic States

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General expressions for quantum electrodynamic corrections to the one-loop self-energy [of order $\alpha(Z\alpha)^6$] and for the two-loop Lamb shift [of order $\alpha^2(Z\alpha)^6$] are derived. The latter includes all diagrams with closed fermion loops. The general results are valid for arbitrary excited non-S hydrogenic states and for the normalized Lamb shift difference of S states, defined as $\Delta_n = n^3\Delta E(nS) - \Delta E(1S)$. We present numerical results for one-loop and two-loop corrections for excited S, P, and D states. In particular, the normalized Lamb shift difference of S states is calculated with an uncertainty of order 0.1 kHz.

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The theory of quantum electrodynamics, when applied to the hydrogen atom and combined with accurate measurements [1,2], leads to the most accurately determined physical constants today [3] and to accurate predictions for transition frequencies. Of crucial importance are higher-order corrections to the bound-state energies, which involve both purely relativistic atomic-physics effects and are mixed with the quantum electrodynamic (QED) corrections. In general, this leads to a double expansion for the energy shifts, both in terms of the QED coupling α (the fine-structure constant) and the nuclear charge number Z.

As is well known, the leading one-loop energy shifts (due to self-energy and vacuum polarization) in hydrogen-like systems are of order $\alpha(Z\alpha)^4$ in units of the electron mass. Analytic calculations for higher excited states in the order $\alpha(Z\alpha)^6$ are extremely demanding. For non-S states, the $\alpha(Z\alpha)^6$ corrections have been obtained recently [4]. However, excited S states are very important for spectroscopy, and the corresponding gap in our knowledge is filled in the current Letter (see Table I). Regarding the two-loop correction, complete results for the $\alpha^2(Z\alpha)^4$ effect were obtained in 1970 (see Ref. [5]). Here, we derive general expressions which allow the determination of the entire two-loop $\alpha^2(Z\alpha)^6$ correction, for all non-S hydrogenic states and the normalized difference $\Delta_n \equiv n^3\Delta E(nS) - \Delta E(1S)$, including the nonlogarithmic term. Together with other available analytic [6,7] and numerical calculations for the $1S$ state [8], our results allow for a much improved understanding of the higher-order two-loop corrections for general excited hydrogenic states, and pave the way for an improved determination of fundamental constants from hydrogen spectroscopy.

The one-loop bound-state self-energy, for the states under investigation here, can be written as

$$\delta^{(1)}E = \frac{\alpha(Z\alpha)^4}{\pi n^3}\{A_{40} + (Z\alpha)^2[A_{61}\ln[(Z\alpha)^{-2}] + A_{60}]\},$$

where the indices of the coefficients indicate the power of $Z\alpha$ and the power of the logarithm, respectively. We work in $D = 4 - 2\epsilon$ spacetime dimensions, and the dimension of space is $d = 3 - 2\epsilon$. Units are chosen so that $\hbar = c = \epsilon_0 = 1$, and the electron mass is unity. A nonrelativistic, "Bethe-style" [9] calculation of the contribution due to ultrasoft photons, in the dipole approximation, leads to a dimensionally regularized energy shift E_{L0},

$$E_{L0} = -\frac{4\alpha}{3\pi}\frac{(Z\alpha)^4}{n^3}\ln k_0 + Z\alpha^2\left\{\frac{2}{3\varepsilon} + \frac{10}{9} + \frac{4}{3}\ln[(Z\alpha)^{-2}]\right\} \times \langle\delta^d(r)\rangle, \tag{1}$$

where $\ln k_0 = \frac{n^3}{2(Z\alpha)^4}\langle p^i(H-E)\ln[2|H-E|/(Z\alpha)^2]p^i\rangle$ is the Bethe logarithm, and $\delta^d(r) = \vec{\nabla}^2 V/(4\pi)$ is a d-dimensional Dirac delta function obtained via the action of the Laplacian on the d-dimensional Coulomb potential $V(r) = -Z\alpha r^{2-d}[\Gamma(\frac{d}{2}-1)\pi^{1-d/2}]$. All matrix elements $\langle\cdot\rangle$ are to be evaluated with regard to the reference state, as given by a nonrelativistic (Schrödinger-Pauli) wave function, and the summation convention is used throughout this Letter.

Following [4,10,11], we now consider corrections due to the relativistic Hamiltonian, the quadrupole term, and the relativistic and retardation corrections to the current. The relativistic correction to the Hamiltonian is

$$H_R = -\frac{\vec{p}^4}{8} + \frac{\pi}{2}Z\alpha\delta^d(r) + \frac{1}{4}\sigma^{ij}\nabla^i V p^j. \tag{2}$$

Here, $\sigma^{ij} \equiv \frac{1}{2i}[\sigma^i, \sigma^j]$. The resulting, dimensionally regularized, correction to the Bethe logarithm is

TABLE I. Values of the nonlogarithmic self-energy correction A_{60} ("relativistic Bethe logarithm") for higher excited S states.

n	$A_{60}(nS)$	n	$A_{60}(nS)$
1	−30.924 149 46(1)	5	−31.455 393(1)
2	−31.840 465 09(1)	6	−31.375 130(1)
3	−31.702 501(1)	7	−31.313 224(1)
4	−31.561 922(1)	8	−31.264 257(1)

$$E_{L1} = \frac{\alpha}{\pi}\frac{(Z\alpha)^6}{n^3}\beta_1 + \frac{\alpha}{3\pi}\left\{\frac{1}{2\varepsilon} + \frac{5}{6} + L(Z\alpha)\right\} \times \left\langle \frac{1}{8}\vec{\nabla}^4 V + \frac{\mathrm{i}}{4}\sigma^{ij}p^i\vec{\nabla}^2 V p^j + 2H_R\bar{G}\vec{\nabla}^2 V\right\rangle, \quad (3)$$

where $L(Z\alpha) \equiv \ln[\frac{1}{2}(Z\alpha)^{-2}]$, and $\bar{G} = 1/(E-H)'$ is the reduced Green function; β_1 is a generalized Bethe logarithm,

$$\frac{(Z\alpha)^6}{n^3}\beta_1 = -\frac{4}{3}\left\langle H_R\bar{G}p^i(H-E)\ln\left[\frac{|H-E|}{(Z\alpha)^2}\right]p^i\right\rangle + \frac{2}{3}\sum_{n,m}\frac{\langle\phi|p^i|n\rangle\langle n|H_R|m\rangle\langle m|p^i|\phi\rangle}{E_m - E_n} \times \left\{(E_n - E)\ln\left[\frac{|E_n - E|}{(Z\alpha)^2}\right] - (E_m - E) \times \ln\left[\frac{|E_m - E|}{(Z\alpha)^2}\right]\right\} + \frac{2}{3}\langle H_R\rangle \times \left\langle p^i\left\{1 + \ln\left[\frac{|H-E|}{(Z\alpha)^2}\right]\right\}p^i\right\rangle. \quad (4)$$

We temporarily restore the reference state ϕ in the notation of the matrix element, and the sums over n and m include both the discrete as well as the continuous part of the spectrum. The argument of the logarithm in β_1 is $\ln[|H-E|/(Z\alpha)^2]$, not $\ln[2|H-E|/(Z\alpha)^2]$ as in $\ln k_0$, and this fact is important for the precise definition of β_1, and of all other generalized Bethe logarithms in the following.

In the dimensional scheme, the quadrupole correction E_{L2}, which was denoted as $F_{\rm nq}$ in former work [10,11], is found to be expressible as $E_{L2} = \mathcal{D}_2 + \mathcal{F}_2$, where

$$\mathcal{D}_2 = \frac{\alpha}{\pi}\left\langle\frac{2(\vec{\nabla}V)^2}{3}\right\rangle\left[\frac{1}{\varepsilon} + \frac{103}{60} + 2L(Z\alpha)\right] + \left\langle\frac{\vec{\nabla}^4 V}{40}\right\rangle\left[\frac{1}{\varepsilon} + \frac{12}{5} + 2L(Z\alpha)\right] + \left\langle\frac{\vec{\nabla}^2 V\vec{p}^2}{6}\right\rangle\left[\frac{1}{\varepsilon} + \frac{34}{15} + 2L(Z\alpha)\right],$$

and $\mathcal{F}_2$ contains the generalized Bethe logarithm β_2,

$$\mathcal{F}_2 = \frac{\alpha(Z\alpha)^6\beta_2}{\pi n^3} = \frac{\alpha}{\pi}\int\frac{d\Omega_{\vec{n}}}{4\pi}(\delta^{ij} - n^i n^j) \times \left\{\left\langle p^i(\vec{n}\cdot\vec{r})^2(H-E)^3\ln\left[\frac{|H-E|}{(Z\alpha)^2}\right]p^j\right\rangle - \left\langle p^i(\vec{n}\cdot\vec{r})(H-E)^3\ln\left[\frac{|H-E|}{(Z\alpha)^2}\right]p^j(\vec{n}\cdot\vec{r})\right\rangle\right\}.$$

Here, $\vec{n}$ is a three-dimensional unit vector, and we integrate over the entire solid angle $\Omega_{\vec{n}}$. Throughout this Letter, $\vec{\nabla}^2$ and $\vec{\nabla}^4$ are understood to exclusively act on the quantity immediately following the operator, i.e., $\langle\vec{\nabla}^2 V\vec{p}^2\rangle = \langle(\vec{\nabla}^2 V)\vec{p}^2\rangle$, $\langle\vec{\nabla}^2 V\overline{G}H_R\rangle = \langle(\vec{\nabla}^2 V)\overline{G}H_R\rangle$, etc.

The correction E_{L3} to the transition current reads $E_{L3} = \mathcal{D}_3 + \mathcal{F}_3$, where $\mathcal{F}_3 = \alpha(Z\alpha)^6\beta_3/\pi n^3$ contains the generalized Bethe logarithm β_3, and

$$\mathcal{D}_3 = -\frac{\alpha}{\pi}\left[\frac{2}{3\varepsilon} + \frac{10}{9} + \frac{4}{3}L(Z\alpha)\right]\left\langle\frac{\vec{\nabla}^2 V\vec{p}^2}{4} + \frac{(\vec{\nabla}V)^2}{2}\right\rangle,$$

$$\mathcal{F}_3 = \frac{2\alpha}{3\pi}\left\langle j^i(H-E)\ln\left[\frac{|H-E|}{(Z\alpha)^2}\right]p^i\right\rangle.$$

Here, $j^i = p^i\vec{p}^2 + \frac{1}{2}\sigma^{ij}\nabla^j V$, and $\nabla^i \equiv \partial/\partial r^i$ denotes the derivative with respect to the ith Cartesian coordinate. The divergences (in ε) in the corrections to the Bethe logarithm are compensated by high-energy virtual photons, which in nonrelativistic QED (NRQED) are given by effective operators. From a generalized Dirac equation (see Chap. 7 of Ref. [12]), one easily obtains the effective one-loop potential

$$\delta^{(1)}V = -\frac{1}{6\epsilon}\frac{\alpha}{\pi}\vec{\nabla}^2 V + \frac{\alpha}{4\pi}\sigma^{ij}\nabla^i V p^j, \quad (5)$$

which in leading order gives rise to the correction $\langle\delta^{(1)}V\rangle$. This correction is a contribution to the middle-energy part E_M, which originates from high-energy virtual photons, with electron momenta of order $Z\alpha$. The corrections of relative order $(Z\alpha)^2$ to $\langle\delta^{(1)}V\rangle$ involve relativistic corrections to the wave function and to the operators, and a two-Coulomb-vertex scattering amplitude. The sum is

$$E_M = \langle\delta^{(1)}V\rangle + 2\langle\delta^{(1)}V\bar{G}H_R\rangle + \frac{\alpha}{\pi}\left(\frac{1}{192} - \frac{1}{48\varepsilon}\right)\left\langle\vec{\nabla}^4 V + 2\mathrm{i}\sigma^{ij}p^i\vec{\nabla}^2 V p^j\right\rangle - \frac{\alpha}{32\pi}\left\langle\left\{\vec{p}^2, \vec{\nabla}^2 V + 2\sigma^{ij}\nabla^i V p^j\right\}\right\rangle - \frac{\alpha}{\pi}\left(\frac{11}{240} + \frac{1}{40\varepsilon}\right)\left\langle\vec{\nabla}^4 V\right\rangle + \frac{\alpha}{\pi}\left(\frac{11}{48} - \frac{1}{3\varepsilon}\right)\langle(\vec{\nabla}V)^2\rangle. \quad (6)$$

The complete one-loop result $\delta^{(1)}E = E_{L0} + E_{L1} + E_{L2} + E_{L3} + E_M$ reads

$$\delta^{(1)}E = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\left(\left[\frac{10}{9} + \frac{4}{3}\ln[(Z\alpha)^{-2}]\right]\delta_{l0} - \frac{4}{3}\ln k_0\right) + \frac{\alpha}{4\pi}\langle\sigma^{ij}\nabla^i V p^j\rangle + \frac{\alpha}{\pi}\frac{(Z\alpha)^6}{n^3}(\beta_1 + \beta_2 + \beta_3) + \frac{\alpha}{\pi}\left\{\left(\frac{5}{9} + \frac{2}{3}L(Z\alpha)\right)\langle\vec{\nabla}^2 V\bar{G}H_R\rangle + \frac{1}{2}\langle\sigma^{ij}\nabla^i V p^j\bar{G}H_R\rangle + \left(\frac{779}{14\,400} + \frac{11}{120}L(Z\alpha)\right)\langle\vec{\nabla}^4 V\rangle + \left(\frac{23}{576} + \frac{1}{24}L(Z\alpha)\right)\langle 2\mathrm{i}\sigma^{ij}p^i\vec{\nabla}^2 V p^j\rangle + \left(\frac{589}{720} + \frac{2}{3}L(Z\alpha)\right)\langle(\vec{\nabla}V)^2\rangle + \frac{3}{80}\langle\vec{p}^2\vec{\nabla}^2 V\rangle - \frac{1}{8}\langle\vec{p}^2\sigma^{ij}\nabla^i V p^j\rangle\right\}. \quad (7)$$

The matrix elements in this result can be evaluated using standard techniques. In terms of the notation of Ref. [4], we have $\mathcal{L} = \sum_{i=1}^{3} \beta_i$. Our general result (7), evaluated for hydrogenic states, reproduces the known logarithmic term A_{61}, and is consistent with all formulas reported for the nonlogarithmic term in Eqs. (10) and (12) of Ref. [4]. The evaluation of $\mathcal{L}$ is a demanding numerical calculation, and numerical values for non-S states have been presented in Table I of Ref. [4]. Taking advantage of the result [10] for $1S$ and the validity of Eq. (7) for the nS $-$ $1S$ difference, we can now proceed to indicate results for the nonlogarithmic term A_{60} for nS states, an evaluation made possible by our generalized NRQED approach (see Table I).

A generalization of our NRQED approach leads to the following general result for the $\alpha^2(Z\alpha)^6$ term of the complete two-loop Lamb shift (including all diagrams with closed fermion loops, see Fig. 1),

$$\begin{aligned}\delta^{(2)}E &= \frac{\alpha^2(Z\alpha)^6}{\pi^2 n^3}\{B_{62}\ln^2[(Z\alpha)^{-2}] + B_{61}\ln[(Z\alpha)^{-2}] + B_{60}\}\\ &= \frac{\alpha^2(Z\alpha)^6}{\pi^2 n^3}\Big\{b_L + \beta_4 + \beta_5 + \Big[\frac{38}{45} + \frac{4}{3}L(Z\alpha)\Big]N\Big\} + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{42923}{259200} + \frac{9}{16}\zeta(2)\ln(2) - \frac{5}{36}\zeta(2) - \frac{9}{64}\zeta(3) + \frac{19}{135}L(Z\alpha)\\ &\quad + \frac{1}{9}L^2(Z\alpha)\Big]\langle\vec{\nabla}^2 V\bar{G}\vec{\nabla}^2 V\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{2179}{10368} - \frac{9}{16}\zeta(2)\ln(2) + \frac{5}{36}\zeta(2) + \frac{9}{64}\zeta(3)\Big]\langle\vec{\nabla}^2 V\bar{G}\vec{p}^4\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{197}{1152} + \frac{3}{8}\zeta(2)\\ &\quad \times\ln(2) - \frac{1}{16}\zeta(2) - \frac{3}{32}\zeta(3)\Big]\langle\vec{p}^4\bar{G}\sigma^{ij}\nabla^i V p^j\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{233}{576} - \frac{3}{4}\zeta(2)\ln(2) + \frac{1}{8}\zeta(2) + \frac{3}{16}\zeta(3)\Big]\langle\sigma^{ij}\nabla^i V p^j\bar{G}\sigma^{ij}\nabla^i V p^j\rangle\\ &\quad + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{197}{2304} + \frac{3}{16}\zeta(2)\ln(2) - \frac{1}{32}\zeta(2) - \frac{3}{64}\zeta(3)\Big]\langle\{\vec{p}^2, \vec{\nabla}^2 V + 2\sigma^{ij}\nabla^i V p^j\}\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{83}{1152} + \frac{17}{8}\zeta(2)\ln(2)\\ &\quad - \frac{59}{72}\zeta(2) - \frac{17}{32}\zeta(3)\Big]\langle(\vec{\nabla}V)^2\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{87697}{345600} + \frac{9}{10}\zeta(2)\ln(2) - \frac{2167}{9600}\zeta(2) - \frac{9}{40}\zeta(3) + \frac{19}{270}L(Z\alpha) + \frac{1}{18}L^2(Z\alpha)\Big]\\ &\quad \times\langle\vec{\nabla}^4 V\rangle + \Big(\frac{\alpha}{\pi}\Big)^2\Big[-\frac{16841}{207360} - \frac{1}{5}\zeta(2)\ln(2) + \frac{223}{2880}\zeta(2) + \frac{1}{20}\zeta(3) + \frac{1}{24}L(Z\alpha)\Big]\langle 2\mathrm{i}\sigma^{ij}p^i\vec{\nabla}^2 V p^j\rangle. \end{aligned} \quad (8)$$

The leading $\alpha^2(Z\alpha)^4$ term, given by the B_{40} coefficient, is well known and therefore not included here (for a review see, e.g., Appendix A of Ref. [3]). The above expression is valid for the P, D states, and for the normalized difference Δ_n of S states. The quantity N is defined in terms of the notation adopted in Refs. [6,13], and the two-loop Bethe logarithm b_L is defined in Refs. [7,14]. Although b_L has been determined numerically only for S states (see Ref. [14]), it represents a well-defined quantity for all hydrogenic states. The logarithmic sum β_4 is given by Eq. (4), with the replacement $H_R \to \frac{1}{4}\sigma^{ij}\nabla^i V p^j$. Finally, we have

$$\frac{(Z\alpha)^6}{n^3}\beta_5 = \frac{1}{2}\Big\langle\sigma^{ij}\nabla^j(H-E)\ln\Big[\frac{|H-E|}{(Z\alpha)^2}\Big]p^i\Big\rangle. \quad (9)$$

Evaluating the general expression (8) for P states, we confirm that $B_{62}(nP) = \frac{4}{27}\frac{n^2-1}{n^2}$. Furthermore, we obtain the results

$$B_{61}(nP_{1/2}) = \frac{4}{3}N(nP) + \frac{n^2-1}{n^2}\Big(\frac{166}{405} - \frac{8\ln 2}{27}\Big), \qquad B_{61}(nP_{3/2}) = \frac{4}{3}N(nP) + \frac{n^2-1}{n^2}\Big(\frac{31}{405} - \frac{8\ln 2}{27}\Big). \quad (10)$$

Numerical values for $N(nP)$ can be found in Eq. (17) of [13]. Regarding the nonlogarithmic term B_{60}, we fully confirm results for the fine-structure difference of P states [15]. A further important conclusion to be drawn from Eq. (8) is that all logarithmic two-loop terms of order $\alpha^2(Z\alpha)^6$ vanish for states with orbital angular momentum $l \geq 2$.

We have also verified that the two-loop result (8) is consistent with the normalized S-state difference Δ_n for the logarithmic terms B_{62} and B_{61}, as derived in Ref. [6] (using a completely different method). Evaluating all matrix elements in Eq. (8), we are now in the position to obtain the n dependence of the nonlogarithmic term, which we write as $B_{60}(nS) - B_{60}(1S) = b_L(nS) - b_L(1S) + A(n)$, where $A(n)$ is the additional contribution beyond the n dependence of the two-loop Bethe logarithm. The result for $A(n)$ is

$$\begin{aligned}A(n) &= \Big(\frac{38}{45} - \frac{4}{3}\ln(2)\Big)[N(nS) - N(1S)] - \frac{337043}{129600} - \frac{94261}{21600n} + \frac{902609}{129600n^2} + \Big(\frac{4}{3} - \frac{16}{9n} + \frac{4}{9n^2}\Big)\ln^2(2) + \Big(-\frac{76}{45} + \frac{304}{135n} - \frac{76}{135n^2}\Big)\\ &\quad \times\ln(2) + \Big(-\frac{53}{15} + \frac{35}{2n} - \frac{419}{30n^2}\Big)\zeta(2)\ln(2) + \Big(\frac{28003}{10800} - \frac{11}{2n} + \frac{31397}{10800n^2}\Big)\zeta(2) + \Big(\frac{53}{60} - \frac{35}{8n} + \frac{419}{120n^2}\Big)\zeta(3)\\ &\quad + \Big(\frac{37793}{10800} + \frac{16}{9}\ln^2(2) - \frac{304}{135}\ln(2) + 8\zeta(2)\ln(2) - \frac{13}{3}\zeta(2) - 2\zeta(3)\Big)[\gamma + \Psi(n) - \ln(n)]. \end{aligned} \quad (11)$$

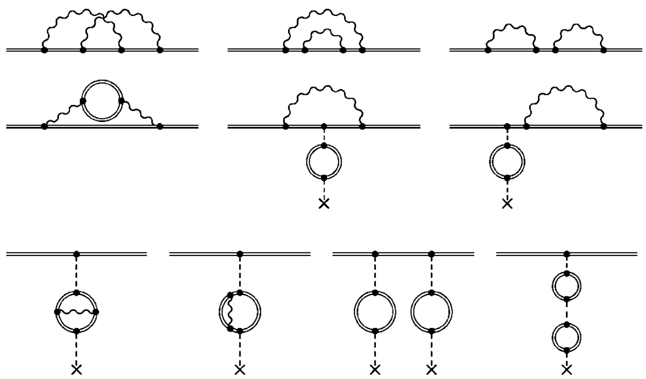

FIG. 1. Two-loop Feynman diagrams for the Lamb shift. The bound-electron propagator is denoted by a double line.

Numerically, $A(n)$ is found to be much smaller than $b_L(nS) - b_L(1S)$, which implies that the main contribution to $B_{60}(nS) - B_{60}(1S)$ is exclusively due to the two-loop Bethe logarithm. As an example, we consider $A(5) = 0.370\,042$ and $B_{60}(5S) - B_{60}(1S) = 21.2(1.1)$, where the error is due to the numerical uncertainty of the two-loop Bethe logarithm $b_L(5S)$ (see Ref. [14]).

The test of standard model theories and the determination of fundamental constants (specifically, of the Rydberg constant and of the electron mass) provide the main motivations for carrying out the QED calculations in ever higher orders of approximation. Recently, our knowledge of the ground-state Lamb shift has been improved by a fully numerical calculation of the two-loop self-energy [8]. However, because of the structure of the hydrogen spectrum, the decisive quantity for the determination of the Rydberg constant from spectroscopic data is the normalized difference Δ_n of the $nS - 1S$ Lamb shift. Elucidating discussions regarding the latter point can be found near Eqs. (2) and (3) of Ref. [16], and in Appendix A of Ref. [3]. Accurate theoretical values for Δ_n can be inferred from the results reported here and are compiled in Table II. The Rydberg constant is currently known to a relative accuracy of 6.6×10^{-12}, limited essentially by the experimental accuracy of the $2S - 8D$ and $2S - 12D$ measurements (see Table V of [3]). Using the improved theory as presented in this Letter, it will become possible to determine the Rydberg constant to an accuracy on the level of 10^{-14}, provided the ongoing experiments concerning the hydrogen $1S - 3S$ transition [17,18] reach a sub-kHz level of accuracy.

TABLE II. Theoretical values of the normalized Lamb-shift difference $\Delta_n = n^3 \Delta E(nS) - \Delta E(1S)$, based on the results reported in this Letter [see Eq. (11)]. Units are kHz.

n	Δ_n	n	Δ_n
2	187 225.70(5)	12	279 988.60(10)
3	235 070.90(7)	13	280 529.77(10)
4	254 419.32(8)	14	280 962.77(10)
5	264 154.03(9)	15	281 314.61(10)
6	269 738.49(9)	16	281 604.34(11)
7	273 237.83(9)	17	281 845.77(11)
8	275 574.90(10)	18	282 049.05(11)
9	277 212.89(10)	19	282 221.81(11)
10	278 405.21(10)	20	282 369.85(11)
11	279 300.01(10)	21	282 497.67(11)

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