Fermions without Fermion Fields

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It is shown that an arbitrary fermion hopping Hamiltonian can be mapped into a system with no fermion fields, generalizing an earlier model of Levin and Wen. All operators in the Hamiltonian of the resulting description commute (rather than anticommute) when acting at different sites, despite the system having excitations obeying Fermi statistics. While extra conserved degrees of freedom are introduced, they are all locally identified in the representation obtained. The same methods apply to Majorana (half) fermions, which for Cartesian lattices mitigate the fermion doubling problem. The generality of these results suggests that the observation of Fermion excitations in nature does not demand that anticommuting Fermion fields be fundamental.

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As fundamental entities, fermion fields appear in conflict with the principle of locality: all fermion creation and annihilation operators *anticommute* no matter how far apart are the points in space at which they act (without restriction by causal connection). This feature is built into quantum field theory through the use of anticommuting Grassman fields, and supersymmetric string theories likewise have explicit anticommuting coordinates. Locality is only preserved in the physics by conservation of fermion number, forcing the fermion operators to appear in pairs. In recent years there has been sustained interest in how particle statistics can be manipulated [1–3], and understanding the fractional quantum hall effect [4] has widened the appreciation that the statistics of the elementary excitations of a system need not simply reflect the statistics of its components. Thus we ask whether it is really necessary to put fermion fields into physics "by hand," or whether they can always be understood as excitations emerging from quantum systems built of operators whose action is strictly local.

It is well known how fermions relate to hard-core bosons in one dimension [5] and how they can be built out of bosons in two dimensions with attached magnetic flux [6], and that neither approach extends naturally to three dimensions of space. The present Letter builds on the recent work of Levin and Wen [1,2], who showed that particular models of pairwise fermion hopping could be represented in terms of operators obeying locality, while the elementary excitations remained strictly fermionic. Their mechanism is essentially the reverse of how Kitaev [7] showed that a particular hexagonal lattice spin model has fermionic excitations. They showed that their mechanism worked on square and cubic lattices explicitly, and interpret it in terms of string-net condensation [2,8].

Here I show that fermion hopping on an *arbitrary* graph can be mapped into the excitations of a Hamiltonian devoid of explicit fermionic operators, in which all operators acting at different sites commute. The dimensionality of space which the graph might approximate is irrelevant to the mapping, which is sensitive only to local coordination numbers (presumed finite). This strongly suggests that quite arbitrary fermion excitation spectra can be represented, and perhaps understood, as arising from the excitation of systems whose fundamental operators obey locality in their commutation properties.

We start from a generic fermion hopping Hamiltonian,

$$H_{\mathrm{hop}} = \sum_i c_i^+ V_i c_i + \sum_{\langle ij\rangle} c_i^+ t_{ij} c_j, \tag{1}$$

where the fermion field c_i has standard anticommutation properties $\{c_i, c_j\} = \{c_i^+, c_j^+\} = 0$ and $\{c_i^+, c_j\} = \delta_{ij}$, the V_i are simple "on-site" potentials (relative to the fermion chemical potential), and the $t_{ij} = t_{ji}^*$ are simple "intersite" hopping matrix elements. The connectivity of the graph (or lattice) is encoded by which elements t_{ij} are nonzero, but below we will need to explicitly restrict the sum over links $\langle ij\rangle$ to those cases. We will also exploit the gauge invariance of the Hamiltonian, that two models related by $t_{jk}^{(2)} = t_{jk}^{(1)} e^{i\vartheta_{jk}}$ are equivalent (through adjustment of phase of the c_k) provided the relative phase factors multiply to unity around all closed loops.

We now introduce new operators $S_{ij} = -S_{ji}$ modulating the hoppings across each link, such that these operators commute with each other and the original fermions, and with eigenvalues $s_{ij} = -s_{ji} = \pm 1$. The Hamiltonian is then generalized to

$$H_{\mathrm{gauge}} = \sum_i c_i^+ V_i C_i - \sum_{\langle ij\rangle} i S_{ij} c_i^+ u_{ij} c_j + \sum_{\langle ij..z\rangle} g_{ij..z} S_{ij} S_{j.}..S_{.z} S_{zi}. \tag{2}$$

Here $u_{ij} = i s_{ij}^0 t_{ij}$ so that the first two terms recover the hopping Hamiltonian (1) for a particular set of outcomes s_{ij}^0 of the operators S_{ij}. We add extra couplings $g_{ij\ldots z}$ to (products round) "Wilson loops" of the S_{ij} operators; setting $g_{ij..z}/(s_{ij}^0 s_{j.}^0..s_{.z}^0 s_{zi}^0)$ sufficiently negative ensures that only combinations of the S_{ij} eigenvalues which are gauge

transformations of the original hopping Hamiltonian contribute to the low energy states of the new Hamiltonian. Because the operators S_{ij} commute with each other and the Hamiltonian, they could be separately diagonalized to their eigenvalues s_{ij}, which are constants of the motion, and we then recover the original tight binding Hamiltonian (up to gauge symmetry). Thus the system does still have its original fermion excitations.

The key to obtaining a representation with locality is now to factorize each link operator into a pair of Majorana (half) fermions,

$$S_{jk} = i m_{jk} m_{kj},$$

where the Majorana half-fermion operators are Hermitian (for simplicity) and have anticommutation properties $\{m_{jk}, m_{j',k'}\} = 2\delta_{jj'}\delta_{kk'}$ as well as *anticommuting* with all the original standard fermion operators c_i. We then associate each new half fermion with the site of its first index, motivating us to rewrite the hopping terms in the Hamiltonian as $b^+_{ij} u_{ij} b_{ji}$, where

$$b_{ij} = m_{ij} c_i.$$

Crucially the new operators b_{ij} commute, $[b_{ij}, b_{i'j'}] = 0$ and $[b^+_{ij,} b_{i'j'}] = 0$, when their left (or site) indices are unequal, $i \neq i'$. The loop terms can also be expressed in terms of operators conforming to locality in this way. Regrouping the factors in each loop gives us $S_{ij}S_{jk}S_{k.}..S_{.z}S_{zi} = B_{i,zj}B_{j,ik}B_{k,j.}..B_{z,.i}$, where

$$B_{i,ik} = i m_{ji} m_{jk}.$$

We can finally eliminate all fermionic notation by writing $n_i = c^+_i c_i$ and hence express the Hamiltonian as

$$H_{\text{gauge}} = \sum_i V_i n_i + \sum_{\langle ij \rangle} b^+_{ij} u_{ij} b_{ji} + \sum_{\langle ij..z \rangle} g_{ij..z} B_{i,zj} B_{j,i}..B_{z,.i}. \tag{3}$$

From their definitions it is trivial to check that any pair drawn from all the operators n_i, b_{ij}, b^+_{ij}, $B_{i,jk}$ appearing in the Hamiltonian (3) commute when their first indices are distinct. As a result we can factor the overall Hilbert space (of wave functions) on which they act into a product of single site Hilbert spaces, and the only nontrivial action of each operator is within the corresponding single site Hilbert space.

We now focus on each site separately and note how the required commutation properties can be explicitly constructed. Each Dirac fermion can be split into a pair of Hermitian Majorana half fermions,

$$2c_k = m_{k0} + i m_{k-1}, \tag{4}$$

in terms of which $n_k = (i m_{k0} m_{k-1} + 1)/2$. The complete set of operator properties then required on-site k are now just the Majorana anticommutation relations

$$\{m_{km}, m_{kn}\} = 2\delta_{mn}, \qquad -1 \leq m, n \leq z_k,$$

where z_k is the coordination number of (i.e., number of links to) site k. These relations are obeyed by standard (Euclidean) 4×4 Dirac matrices for $2 + z_k \leq 5$ and by their $2^s \times 2^s$ generalizations for $2 + z_k \leq 2s + 1$. It is crucial that we do *not* require to represent fermion anticommutation between local fermion operators on *different* sites, because all the terms in the Hamiltonian contain an even number of fermion factors from each site. Writing $\gamma_k(i)$ for the k'th Dirac matrix acting in the Hilbert space of the i'th site, we then have all the operators in the bosonic Hamiltonian (3) expressed in terms of these:

$$n_j = \frac{i}{2}\gamma_0(j)\gamma_{-1}(j) + \frac{1}{2},$$

$$b_{jk} = \frac{1}{2}\gamma_{k'}(j)[\gamma_0(j) + i\gamma_{-1}(j)], \qquad B_{j,ik} = i\gamma_{i'}(j)\gamma_{k'}(j). \tag{5}$$

Here on-site j, $1 \leq k'(j,k) \leq z_j$ denotes the local index associated with its link to site k.

The end result is that all the operators appearing in the Hamiltonian are expressed in terms of local matrix operators, which in turn are all equivalent to bilinear combinations of Dirac matrices: these might loosely be termed "spins." Where the original hopping Hamiltonian had links with explicit fermion hopping, the derived generalization of Levin and Wen's model [1] has coupling between the local spins. Most importantly, the spin operators for different sites commute—yet by construction the system still has the original fermionic excitations.

The loop products of operators $B_{i,jk}$ can be further simplified in terms of products of new operators

$$P_{ij} = \gamma_{j'(i,j)}(i)\gamma_{i'(j,i)}(j) \tag{6}$$

in direct index correspondence with the original operators S_{ij}. These new operators are *not* equivalent to the S_{ij} operators; in particular, two operators P_{ij} *anticommute* if they have one site index in common.

Locality of all operators has been gained at the expense of enlarging the Hilbert space. The original hopping Hamiltonian (1) acted on a Hilbert space of dimension 2^N where N is the number of sites, equivalent to N qubits, one for each site. To construct the generalized Wen Hamiltonian [(3) with (5)] we first added qubits equal to the number of links, A. However, in the final local representation we carried fewer noncommutations, and multiplying the dimension of all the local Hilbert spaces leads to a dimension equivalent to total qubit count

$$C = N_E + \frac{1}{2}N_O + A.$$

Here N_E and N_O are the counts of sites with even and odd coordination number, respectively, the even being less efficiently represented because the number of anticommuting Dirac matrices is naturally odd.

We should now expect that there are $A - \frac{1}{2}N_O$ constants of the motion, and these and one more can be found as

follows. First, from every link ij, we have the corresponding P_{ij} commuting with every term in the Hamiltonian (3). The result is that arbitrary (product) strings of P operators commute provided they have no ends in common, including closed strings which have no ends. Second, for every even coordinated site we have Hermitian $\Gamma_k = i^{(z+2)/2}\prod_{j=-1}^{z}\gamma_j(k)$ anticommuting with every γ on that site, and hence commuting with every term in the Hamiltonian; however, Γ_k anticommutes with any P-string ending on site k. The maximal commuting set of all these operators then appears to be the union of: (a) all mutually inequivalent closed loop P strings, equivalent to a minimal set of independent loop terms in the Hamiltonian (3), numbering $A - N + 1$; (b) the Γ_k for all even sites, numbering N_E; (c) open P strings ending on odd sites, with no ends in common and inequivalent under products with closed loop strings, all of which enumerate to $N_O/2$ by taking all the odd sites in (arbitrary) disjoint pairs.

All of the above commute with each other and with the Hamiltonian, so we have in total $A - \frac{1}{2}N_O + 1$ explicit constants of the motion. Each corresponding operator has eigenvalues ± 1, so its conservation removes one qubit from the dynamics. The author conjectures that the one extra conservation law relates to conservation of fermion number (modulo 2).

Can one exploit the constants of the motion to reduce the size of the (quantum-mechanically coupled) Hilbert space? Trying this with P_{ij} operators induces successively less local anticommutation relations, tending to rebuild the original fermionic representation. However, we can eliminate the sitewise local Γ_i on even sites, leading us to a dimension-independent generalization of what in one dimension corresponds to the anisotropic Heisenberg model representation of fermions.

Let us focus on some particular even site i and in the following drop reference to that index. We can eliminate $\gamma_{-1} = i^{(z+2)/2}\gamma_0\gamma_1\ldots\gamma_z\Gamma$, and then the operators in the Hamilonian which contained γ_{-1} take the forms $n = -\frac{i^{z/2}}{2}\gamma_1\ldots\gamma_z\Gamma + 1/2$, $b_k = \frac{1}{2}\gamma_{k'}\gamma_0(1 - i^{z/2}\gamma_1\ldots\gamma_z\Gamma)$. Then because Γ commutes with the Hamiltonian we can focus on the sector with eigenvalue $\Gamma = 1$ and make this replacement. Now the only matrices appearing are γ_k, $k = 0\ldots z$, and we can take a minimal anticommuting representation of these [9], in terms of which we obtain $n = \frac{1}{2}\times(1 + \gamma_0)$, $b_k = \gamma_{k'}n$, $b_k^+ = n\gamma_{k'} = \gamma_{k'}(1 - n)$.

For an arbitrary graph of even coordinated sites, we now have Hamiltonian

$$H_{\text{even}} = \sum_i V_i n_i + \sum_{\langle ij\rangle} n_i\gamma_{j'}(i)u_{ij}\gamma_{i'}(j)n_j + \sum_{\langle ij\ldots z\rangle} g_{ij\ldots z}P_{ij}P_{j\ldots}P_{,z}P_{zi}, \tag{7}$$

and the explicit form for the P_{ij} remains as per Eq. (6), except that the dimension of the local Dirac matrices has been halved. It is now particularly clear how the hopping terms conserve fermion numbers, and indeed one can further show that $n_i\gamma_{j'}(i)u_{ij}\gamma_{i'}(j)n_j = n_i(1 - n_j)\times\gamma_{j'}(i)u_{ij}\gamma_{i'}(j)(1 - n_i)n_j$ making the interconnection between occupation numbers 1_j0_i and 0_j1_i totally explicit. The loop terms are also particle conserving and each P_{ij} factor can be reorganized in similar manner. If one specializes to $z = 2$ corresponding to a one-dimensional chain of sites, the Dirac matrices reduce to Pauli matrices and the first two terms of the Hamiltonian are exactly equivalent to the anisotropic Heisenberg spin chain [10], well known to represent fermions in one dimension. It is gratifying that such a long-known fermion Hamiltonian turns out to have natural extension to arbitrary dimensions and even to arbitrary graphs (of even coordination number).

All of our analyses can be generalized to the case where we start from hopping of an arbitrary number h of half fermions on each site. Hitherto we started from one standard fermion per site in the fermion hopping Hamiltonian (1) and split that into two half fermions (4), $h = 2$. In the general case any natural number h is allowed, with the on-site potentials and hopping matrix elements making arbitrary (Hermitian) mixings amongst the h components. For $h = 1$ there are no on-site potential terms, and hermiticity restricts the hopping matrix elements to be pure real, $u_{ij} = u'_{ij}$, giving us

$$H_{\text{hop},1} = i\sum_{\langle ij\rangle} u'_{ij}m_{0i}m_{0j}. \tag{8}$$

The standard fermion case discussed earlier is $h = 2$, and reduces to the sum of two (commuting) $h = 1$ Hamiltonians if the on-site potentials are zero and the u_{ij} are all pure real. The case $h = 3$ commands interest for ''generations'' in particle physics.

The discussion of conservation laws and reduction of Hilbert space generalizes quite trivially, with the understanding that *even* and *odd* sites are identified by whether $h + z$ is even or odd. Particularly simple spin models are obtained from $h = 1$, where the local operators b_{ij} are Hermitian and can be directly represented as single local Dirac matrices $\gamma_j(i)$ (rather than bilinear products) requiring only z_i Dirac matrices at each site, and with $B_{i,jk} \propto i\gamma_j(i)\gamma_k(i)$. Quite general Majorana fermion hopping Hamiltonians can therefore emerge from the excitations of models built out of simple local spin operators. The gauge symmetry changes naturally with h. For $h = 1$ we have only $Z2$ (sign changes), corresponding to the transformations exploited in the ''KLW trick'' of introducing the S operators, and for $h = 2$ we already noted $U(1) = O(2)$ gauge symmetry. For $h = 3$ we would have gauge symmetry $O(3)$ if the on-site terms are sufficiently symmetric, but quite arbitrary mixing of components allowed in the on-site terms of the Hamiltonian would in general reduce the gauge symmetry to $O(2)$.

The simple (hyper-)cubic lattices are of special interest as a source of insight into the continuum limit, and particle

physics interest focuses particularly on the case where the fermion excitations are (almost) massless. The special case of the main analysis above previously presented by Levin and Wen, which was for simple square and cubic lattices with $h = 2$, falls into this category. However, that work also suffered from the standard ''Fermion doubling problem'' of lattice fermion models [11], yielding 2^d massless fermions in d dimensions.

Starting from a single half-fermion ($h = 1$) hopping model halves the fermion doubling to give just 2^{d-1} massless fermions in d dimensions. From the point of view of the spin models, this is just as natural as $h = 2$. For simplicity we present the calculation in one dimension, which suffices to demonstrate the novelty, as the elaboration to higher dimensions simply follows in the same manner as Levin and Wen [1,2]. A simplest one-dimensional Majorana hopping Hamiltonian is

$$H_1 = -iu' \sum_n m_n m_{n+1}$$

and imposing periodic boundary conditions for such a system of N sites results in a spectrum of standard massless fermions,

$$H_1 = 2u' \sum_{k=0}^{\pi} (2\tilde{c}_k^+ \tilde{c}_k - 1) \sin k.$$

Here the wave vectors span *half* the first Brillouin zone of the lattice, $k = n2\pi/N$, $n = 1$ to $N/2$, the fermion operators are given by $\tilde{c}_k = 1/\sqrt{2N}\sum_n e^{-ikn} m_n$, and the negative wave vector Fourier components give their conjugates. These $\tilde{c}_k$ are standard full fermion operators, in particular, obeying $\{\tilde{c}_k^+, \tilde{c}_l\} = \delta_{kl}$.

The above is in effect a staggered fermion [12] solution to the fermion doubling problem [11]. The full fermions obtained by halving the Brillouin zone can be mapped onto one full conserved fermion per two original sites, but these are nonlocally related to the original half fermions. This parallels another solution to the fermion doubling problem, in which similarly nonlocal hopping is used to force a better spectral approximation to the continuum [13].

One can introduce mass without doubling the spectrum, simply by modulating the bond strengths $u'_{n,n+1} \to \sqrt{u'^2 + \epsilon^2} + \epsilon(-1)^n$ to open up an energy gap at $E = 0$ (corresponding to $k = 0, \pi$). One then finds two excitations with, $E(k) = \pm 4\sqrt{u'^2 \sin^2 k + \epsilon^2}$, over the quarter Brilliouin zone $0 \le k < \pi/2$. Adding more Cartesian dimensions with suitably frustrated signs to the couplings leads [as per Levin and Wen [1,2] for the full fermion case] to the natural generalizations in higher dimensions, for example, $E(k_x, k_y, k_z) = \pm 4\sqrt{u_x'^2 \sin^2 k_x + u_y'^2 \sin^2 k_y + u_z'^2 \sin^2 k_z + \epsilon^2}$ in three dimensions. Each increase in dimension generates a doubling of the spectrum to two species or a two component fermion in $d = 2$, and a four component fermion in $d = 3$, matching standard free fermion fields.

In overall summary, we can now construct Hamiltonians in terms of strictly local operators (in the sense of their commutation relations) whose excitations correspond to fermion hopping on whatever graph is desired. If one accepts intersite hopping as an approximation to continuum particle dynamics, then this means that almost any fermion problem can be constructed out of the excitations of what might loosely have been described as a Bose system.

The method has been presented for free (noninteracting) fermion systems. However, it is trivially generalizable to the case of an arbitrary interaction, provided this is a function of the state number operators (or other bilinear combinations of local fermion variables). In particular, Coulomb interactions between different sites are allowed, as are Hubbard interactions. Modifying hopping according to occupation numbers, as in t-J model, is also readily incorporated.

The ''spin'' Hamiltonians constructed to give exactly the specified fermions are somewhat cumbersome. Are there simpler local spin Hamiltonians which still give fermion excitations? The generalized anisotropic Heisenberg Hamiltonian is one case in point: setting the on-site terms to zero and removing the number operator factors from the hopping terms turns out to give the same as starting from a Majorana hopping Hamiltonian (8). Also, we could by deliberate construction build Hamiltonians which are supersymmetric in their excitation spectrum, just by adding in the desired bosons. Are there variations on the theme which are more naturally supersymmetric?

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