The Spectral Dimension of the Universe is Scale Dependent

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We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be ‘‘self-renormalizing’’ at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

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Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory. The outstanding challenge is to construct *a* consistent quantum description of this highly nonperturbative gravitational regime that stands a chance of being physically correct.

Slow progress in the quest for quantum gravity has not hindered speculation on what kind of mechanism may be responsible for resolving the short-distance singularities. A recurrent idea is the existence of a minimal length scale, often in terms of a characteristic Planck-scale unit of length in scenarios where the spacetime at short distances is fundamentally discrete.

The alternative we will advance here is based on new results from an analysis of the properties of quantum universes generated in the nonperturbative and background-independent causal dynamical triangulations (CDT) approach to quantum gravity. As shown in [1,2], they have a number of appealing macroscopic properties: first, their scaling behavior as a function of the spacetime volume is that of genuine isotropic and homogeneous four-dimensional worlds. Second, after integrating out all dynamical variables but the scale factor $a(\tau)$ *in the full quantum theory*, the correlation function between scale factors at different (proper) times τ is described by the simplest minisuperspace model used in quantum cosmology.

We have recently begun an analysis of the *microscopic* properties of these quantum spacetimes. As in previous work, their geometry can be probed in a rather direct manner through Monte Carlo simulations and measurements. At small scales, it exhibits neither fundamental discreteness nor indication of a minimal length scale. Instead, we have found evidence of a fractal structure (see [3], which also contains a detailed technical account of the numerical setup). What we report on in this Letter is a most remarkable finding concerning the universes' *spectral dimension*, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the scale factor. Surprisingly, the spectral dimension turns out to be scale dependent and decreases smoothly from four to a value of around two as the quantum spacetime is probed at ever smaller distances. This suggests a picture of physics at the Planck scale which is radically different from frequently invoked scenarios of fundamental discreteness: through the dynamical generation of a scale-dependent dimensionality, nonperturbative quantum gravity provides an effective ultraviolet cutoff through *dynamical dimensional reduction*.

The spectral dimension.—A diffusion process on a d-dimensional Euclidean geometry with a fixed, smooth metric $g_{ab}(\xi)$ is governed by the diffusion equation

$$\frac{\partial}{\partial \sigma} K_g(\xi, \xi_0; \sigma) = \Delta_g K_g(\xi, \xi_0; \sigma), \tag{1}$$

where σ is a fictitious diffusion time, Δ_g the Laplace operator corresponding to $g_{ab}(\xi)$, and $K_g(\xi, \xi_0; \sigma)$ the probability density of diffusion from ξ to ξ_0 in diffusion time σ. We will consider processes which are initially peaked at some point ξ_0,

$$K_g(\xi, \xi_0; \sigma = 0) = \frac{\delta^d(\xi - \xi_0)}{\sqrt{\det g(\xi)}}. \tag{2}$$

A quantity that is easier to measure than K_g in numerical simulations is the average *return probability*

$$P_g(\sigma) := \frac{1}{V} \int d^d\xi \sqrt{\det g(\xi)} K_g(\xi, \xi; \sigma), \tag{3}$$

where $V = \int d^d\xi \sqrt{\det g(\xi)}$ is the spacetime volume. Note that $P_g(\sigma)$ is a diffeomorphism-invariant quantity.

For an infinite flat space, the solution to Eq. (1) is simply given by

$$K_g(\xi, \xi_0; \sigma) = \frac{e^{-d_g^2(\xi,\xi_0)/4\sigma}}{(4\pi\sigma)^{d/2}}, \qquad g_{ab}(\xi) = \delta_{ab}, \tag{4}$$

where $d_g(\xi, \xi_0)$ denotes the geodesic distance between ξ and ξ_0. It follows that $\sqrt{\sigma}$ is an effective measure of the linear spread of the Gaussian at diffusion time σ. Because of $P_g(\sigma) = 1/\sigma^{d/2}$ in the flat case, we can extract the dimension d of the manifold by taking the logarithmic derivative,

$$-2\frac{d\log P_g(\sigma)}{d\log\sigma} = d, \tag{5}$$

independent of σ.

For curved spacetimes and/or finite spacetime volume V one can still use Eq. (5) to extract the dimension, but there will be corrections for sufficiently large σ. For finite volume, in particular, $P_g(\sigma)$ goes to one for $\sigma \gg V^{2/d}$ since the zero mode of the Laplacian $-\Delta_g$ will dominate the diffusion in this region. For a given diffusion time σ the behavior of $P_g(\sigma)$ is determined by eigenvalues λ_n of $-\Delta_g$ with $\lambda_n \leq 1/\sigma$, and the contribution from higher eigenvalues is exponentially suppressed. Like in the flat case, large σ is related to diffusion which probes spacetime at large scales, whereas small σ probes short distances.

We will use the return probability to determine an effective dimensionality of quantum spacetime, which in our nonperturbative path integral formulation amounts to studying diffusion on an entire *ensemble* of curved, Euclidean(ized) geometries. Since the return probability $P_g(\sigma)$ in (3) is invariant under reparametrizations, one can define the quantum average $P_V(\sigma)$ of $P_g(\sigma)$ over all equivalence classes $[g_{ab}]$ of metrics with a given spacetime volume V by

$$P_V(\sigma) = \frac{1}{Z_V}\int \mathcal{D}[g_{ab}]e^{-S_E(g_{ab})}\delta\left(\int d^4\xi\sqrt{\det g} - V\right)P_g(\sigma), \tag{6}$$

where $S_E(g_{ab})$ is the (Euclidean) Einstein-Hilbert action of $g_{ab}(\xi)$ and Z_V the partition function (path integral) for geometries with fixed volume V. Given the definition (6), $P_V(\sigma)$ is gauge invariant whenever the functional integral over geometries is defined gauge invariantly. We have no *a priori* knowledge of how the functional average (6) will affect corrections to (5) due to fixed nonflat geometries. If the example of two dimensional Euclidean quantum gravity is anything to go by (see [4–7])—as we will assume in what follows—any reference to the curvature terms will average out and the corrections to (5) will be a function of only σ and V.

It is straightforward to generalize the diffusion process and its associated probability density to the piecewise linear geometries that appear in the path integral of CDT [3]. In analogy with the ordinary path integral for a particle, one would expect that in the continuum limit a typical geometry in the quantum ensemble is still continuous, but nowhere differentiable; for example, it could be fractal. In fact, diffusion on fractal structures is well studied in statistical physics [8], and there the return probability takes the form

$$P_N(\sigma) = \sigma^{-D_S/2}F\left(\frac{\sigma}{N^{2/D_S}}\right), \tag{7}$$

where N is the "volume" associated with the fractal structure and D_S is the so-called *spectral dimension*, which is not necessarily an integer. An example of fractal structures are branched polymers, which generically have a spectral dimension $D_S = 4/3$ [4,9]. The function $F(x)$ in (7) goes to 1 for $x \to 0$, i.e., for large N, and falls off at least exponentially for $x > 1$ in the lower-dimensional quantum gravity theories studied so far [4–7].

For the nonperturbative gravitational path integral defined by CDT, we expect the same functional form (7), where N now stands for the discrete four-volume, given in terms of the number of four-simplices of a triangulation. As above in (5), we can now extract the value of the fractal dimension D_S by measuring the logarithmic derivative,

$$D_S(\sigma) = -2\frac{d\log P_N(\sigma)}{d\log\sigma}, \tag{8}$$

as long as the diffusion time σ is not much larger than N^{2/D_S}. From previous numerical simulations of 2D quantum gravity in terms of dynamical triangulations [5–7], we already know that Eq. (8) is not reliable for arbitrary diffusion times. The observed behavior of $D_S(\sigma)$ for a given triangulation will typically exhibit irregularities for the smallest σ, caused by the lattice discretization, and then enter a long and stable regime where the spectral dimension is independent of σ, before finite-size effects start to dominate and $D_S(\sigma)$ goes to zero. [Often the behavior of $P_N(\sigma)$ for odd and even numbers of diffusion steps σ will be quite different for small σ and merge only for $\sigma \approx 20$–30.—The origin of this asymmetry is illustrated by the (extreme) example of diffusion on a one-dimensional piecewise straight space, where the return probability simply vanishes for any odd number σ of steps.]

Measuring the spectral dimension.—In the CDT approach, quantum gravity is defined as the continuum limit of a regularized version of the nonperturbative gravitational path integral [10,11]. The set of spacetime geometries to be summed over is represented by a class of causal four-dimensional piecewise flat manifolds ("triangulations"). Every member T of the ensemble of simplicial spacetimes can be wick rotated to a unique Euclidean piecewise flat geometry, whereupon the path integral assumes the form of a partition function

$$Z = \sum_T \frac{1}{C_T}e^{-S_E(T)}, \tag{9}$$

where C_T is a combinatorial symmetry factor and $S_E(T)$ the Euclidean Einstein-Regge action of the triangulation T. All geometries are assembled from elementary building blocks, so-called four-simplices (four dimensional analogues of flat triangles). They share a global, discrete version of proper time, with respect to which no topology changes of the spatial triangulations are allowed. For an explicit and precise definition of the class of piecewise linear geometries T which appear in the sum (9) we refer to [11]. In the continuum limit, the CDT time τ becomes proportional to the cosmological proper time of a conventional minisuperspace model [2]. (The proper time τ of CDT—although invariantly defined from a geometric point of view—does not necessarily coincide with a physical time appearing explicitly in the formulation of physical observables, for example, two-point functions.)

We extract information about the continuum limit of the theory by Monte Carlo simulations and a finite-size scaling analysis of (9). Further details about the updating moves for the geometry and the numerical setup can be found in [3,11], respectively. For computer-technical reasons we keep the total spacetime volume N approximately fixed during simulations. All the results obtained can be related by an inverse Laplace transform to those for the geometric ensemble with the volume constraint absent.

Our measurements of the spectral dimension were performed in the phase of the statistical model (9) which generates a quantum geometry extended in both space and time, with large-scale four-dimensional scaling properties [1,3]. As discussed in [2], the quantum universe we generate is a "bounce" in the Euclidean sector of the theory. It has a characteristic shape when we plot its three-volume [equivalently, its scale factor $a(\tau)$] as a function of proper time τ. The universe starts out with a minimal three-volume, increases to a maximum, and then decreases in a symmetric fashion back to a minimal value. As shown in [2], the underlying dynamics of the scale factor $a(\tau)$ solves the equation of motion of a simple minisuperspace action for $a(\tau)$, and is that of a bounce, with total Euclidean spacetime volume determined by the volume at which the simulation is performed. It is on an ensemble of geometries of this type that we made our measurements. Since we are interested in the bulk properties of the diffusion, we always started the process from a simplex in the constant-time slice where $a(\tau)$ is maximal. The simulations to determine the universe's spacetime spectral dimension were performed for geometries of proper-time extension $t = 80$ and a discrete volume of up to approximately $N = 181.000$ four-simplices, and the diffusion was followed for up to $\sigma_{\max} = 400$ time steps.

Figure 1 summarizes our measurements of the spectral dimension $D_S(\sigma)$ at the maximal spacetime volume, extracted as the logarithmic derivative (8) from a discrete implementation of the diffusion process. The (envelopes of the) error bars represent the errors coming from averaging over 400 different measurements of diffusion processes, performed for independent starting points and statistically independent configurations T generated by the Monte Carlo simulation. As observed previously in other systems of random geometry, we have found a different behavior of $D_S(\sigma)$ for odd and even (discrete) diffusion times σ for small σ. In order to eliminate this short-distance lattice artifact, we have only included the region $\sigma \geq 40$ for which the odd and even curves coincide, both in Fig. 1 and in determining the spectral dimension.

The data points along the central curve in Fig. 1 represent our best approximation to $D_S(\sigma)$ in the limit of infinite spacetime volume. Their monotonic increase as a function of σ indicates that we have not yet reached the region where finite-volume effects dominate (in the form of the constant mode of the Laplacian). The remarkable feature of the behavior of the spectral dimension illustrated in Fig. 1 is that it is qualitatively different from what has been observed in similar systems up to now, be it in two-dimensional Euclidean quantum gravity with or without matter [5] or for the spatial hypermanifolds of our present CDT setup in four dimensions [3]. In these cases, immediately following the region of even-odd asymmetry for small σ, $D_S(\sigma)$ stabilizes in a horizontal line, indicating the presence of a single value $D_S^{(0)}$ characterizing the spectral dimension of the system, independent of the scale at which the diffusion process probes the geometry.

Apparently, this is not the case for the spectral spacetime dimension in quantum gravity defined by CDT. The measurements shown in Fig. 1 indicate that $D_S(\sigma)$ changes with the scale probed. [The σ dependence of $D_S(\sigma)$ does *not* come from the finite-size function $F(\sigma/N^{2/D_S})$ in (7), which is well approximated by $F \approx 1$ in the σ range considered, as has been checked by studying the diffusion for a variety of different values $N < 181.000$ [3].] In order to quantify this scale dependence we have attempted a variety of fits in the available data range $\sigma \in [40, 400]$. Among curves with three free parameters of the form const. + asymptotic form (The two alternatives considered were $a - be^{-c\sigma}$ and $a - b/\sigma^c$.), a fit of the form

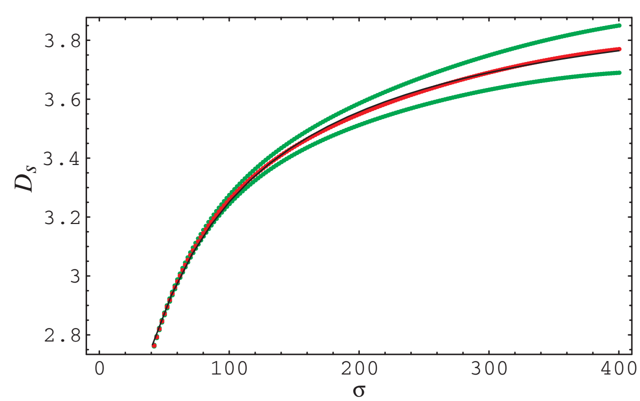

FIG. 1 (color online). The data points along the central curve show the spectral dimension $D_S(\sigma)$ of the universe as a function of the diffusion time σ. Superimposed is a best fit, the continuous curve $D_S(\sigma) = 4.02 - 119/(54 + \sigma)$. The two outer curves quantify the error bars, which increase linearly with σ, due to (8). (Measurements taken for a quantum universe with 181.000 four-simplices.)

$$-2\frac{d\log P(\sigma)}{d\log\sigma} = a - \frac{b}{\sigma + c} \tag{10}$$

agrees best with the data. In Fig. 1, the curve

$$D_S(\sigma) = 4.02 - \frac{119}{54 + \sigma} \tag{11}$$

has been superimposed on the data, where the three constants were determined from the entire data range $\sigma \in [40, 400]$. Although both b and c individually are slightly altered when one varies the range of σ, their ratio b/c as well as the constant a remain fairly stable. Integrating relation (10), we have

$$P(\sigma) \sim \frac{1}{\sigma^{a/2}(1 + c/\sigma)^{b/2c}}, \tag{12}$$

implying a behavior

$$P(\sigma) \sim \begin{cases} \sigma^{-a/2} & \text{for large } \sigma, \\ \sigma^{-(a-b/c)/2} & \text{for small } \sigma. \end{cases} \tag{13}$$

Our interpretation of Eqs. (12) and (13) is that the quantum geometry generated by CDT does not have a self-similar structure at all distances, but instead has a scale-dependent spectral dimension which increases continuously from $a - b/c$ to a with increasing distance.

Taking into account the variation of a in Eq. (10) when using various cuts $[\sigma_{\min}, \sigma_{\max}]$ for the range of σ, as well as different weightings of the errors, we obtain the asymptotic value

$$D_S(\sigma = \infty) = 4.02 \pm 0.1, \tag{14}$$

which means that the spectral dimension extracted from the large-σ behavior (which probes the long-distance structure of spacetime) is compatible with four. On the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

Discussion.—The continuous change of spectral dimension described in this Letter constitutes to our knowledge the first dynamical derivation of a scale-dependent dimension in full quantum gravity. (In the so-called exact renormalization group approach to Euclidean quantum gravity, a similar reduction has been observed recently in an Einstein-Hilbert truncation [12].) It is natural to conjecture it will provide an effective short-distance cutoff by which the nonperturbative formulation of quantum gravity employed here, causal dynamical triangulations, evades the ultraviolet infinities of perturbative quantum gravity. Contrary to current folklore (see [13] for a review), this is done without appealing to short-scale discreteness or abandoning geometric concepts altogether.

Translating our lattice results to a continuum notation requires a "dimensional transmutation" to dimensionful quantities, in accordance with the renormalization of the lattice theory. Because of the perturbative nonrenormalizability of gravity, this is expected to be quite subtle. CDT provides a concrete framework for addressing this issue and we will return to it elsewhere. However, since σ from (1) can be assigned the length dimension two, and since we expect the short-distance behavior of the theory to be governed by the continuum gravitational coupling G_N, it *is* tempting to write the continuum version of (10) as

$$P_V(\sigma) \sim \frac{1}{\sigma^2}\frac{1}{1 + \text{const.} \times G_N/\sigma}, \tag{16}$$

where const. is a constant of order one. Using the same naïve dimensional transmutation, one finds that our "universe" of 181.000 discrete building blocks has a spacetime volume of the order of $(20 l_{\rm Pl})^4$ in terms of the Planck length $l_{\rm Pl}$, and that the diffusion with $\sigma = 400$ steps corresponds to a linear diffusion depth of $20 l_{\rm Pl}$, and is therefore of the same magnitude. The relation (16) describes a universe whose spectral dimension is four on scales large compared to the Planck scale. Below this scale, the quantum-gravitational excitations of geometry lead to a nonperturbative dynamical dimensional reduction to two, a dimensionality where gravity is known to be renormalizable.

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