Nonextensive Boltzmann Equation and Hadronization

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(Received 1 April 2005; published 12 October 2005)

We present a novel nonextensive generalization of the Boltzmann equation. We investigate the evolution of the one-particle distribution in this framework. The stationary solution is exponential in a nonlinear function of the original energy. The total energy is composed using a general, associative nonextensive rule. We propose that for describing the hadronization of quark matter such rules may apply.

DOI: 10.1103/PhysRevLett.95.162302 PACS numbers: 25.75.Nq, 02.70.Ns, 05.20.Dd, 05.90.+m

There is a lot of experimental evidence on power-law tailed statistical distributions of single-particle energy, momenta, or velocity. In particular, hadron transverse momentum spectra at central rapidity, which stem from elementary particle and heavy ion collisions, can be well fitted by a formula reflecting m_T scaling [1–8]: $f(p_T) \sim (1 + m_T/E_c)^{-v}$. Interpreting these spectra as a distribution in the transverse directions at zero rapidity, the single-particle energy is given by $E = m_T = \sqrt{p_T^2 + m^2}$ for a relativistic particle with mass m. Amazingly, this formula describes exactly the Tsallis distribution $f(E)$, which was obtained by using theoretical arguments of thermodynamical nature [9]. Distributions with a power-law-like tail, in particular, the Tsallis distribution, can be seen in many areas where statistical models apply [10–18]. It was investigated as a generic feature in the framework of nonextensive thermodynamics [19–21]. Tsallis has suggested an expression for the entropy, also encountered earlier by others [9,22], which is a generalization of the Boltzmann formula. From this, with a canonical constraint on the total energy, the power-law tailed distribution can be derived. Without being able to exclude a nonequilibrium interpretation of the power-law tail, it is tempting to investigate the possibility that some nonexponential spectra would be a result of a particular form of (meta)equilibrium, featuring characteristics of a nonextensive thermodynamics.

In this Letter we propose a possible way to understand power-law tailed energy distributions as stationary solutions to a generalized two-body Boltzmann equation. We show that this two-body Boltzmann equation allows for nonexponential stationary single-particle distributions, if the two-body distribution *factorizes*, but the two-body energy is *not extensive*. The Tsallis distribution is a special case thereof.

It is a widespread belief that only the exponential distribution can be the stationary solution to the Boltzmann equation, but this statement is true only with a few restrictions: (i) if the two-particle distributions factorize, (ii) the two-particle energies are additive in the single-particle energies ($E_{12} = E_1 + E_2$) and (iii) the collision rate is multilinear in the one-particle densities. A generalization of the original Boltzmann equation has been pioneered by Kaniadakis [23] considering a general, nonlinear density dependence of the collision rates. An "H_q" theorem for the particular Tsallis form of the collision rate has been derived by Lima, Silva, and Plastino [24]. Here we follow another ansatz; we modify the linear Boltzmann equation in the energy balance part only: instead of requiring $E_1 + E_2 = E_3 + E_4$ in a $1 + 2 \leftrightarrow 3 + 4$ two-body collision we consider a general, not necessarily extensive, rule:

$$h(E_1, E_2) = h(E_3, E_4). \tag{1}$$

It is physically sensible to choose the function $h(x, y)$ symmetric and satisfying $h(E, 0) = h(0, E) = E$. Also, for applying the same rule for subsystems combined themselves of subsystems, associativity is required: $h(h(x, y), z) = h(x, h(y, z))$. This way the same rule applies for the elementary two-particle system as for large subsystems in the thermodynamical limit. It is known that the general mathematical solution of the associativity requirement is given by

$$h(x, y) = X^{-1}[X(x) + X(y)], \tag{2}$$

with $X(0) = 0$ and $X(t)$ being a continuous, strict monotonic function [25]. Composing the formula (2) with the function X and taking the partial derivative with respect to y at $y = 0$ one obtains an ordinary differential equation for $X(x)$ with the solution

$$X(E) = X'(0) \int_0^E \frac{dx}{\frac{\partial h}{\partial y}(x, 0)}. \tag{3}$$

Because of $X[h(E_1, E_2)] = X(E_1) + X(E_2)$, the quasi-energy $X(E)$ is an additive quantity and the rule (1) is equivalent to

$$X(E_1) + X(E_2) = X(E_3) + X(E_4). \tag{4}$$

Applying such a general energy composition rule (4), the rate of change of the one-particle distribution is given by

$$\dot{f}_1 = \int_{234} w_{1234}[f_3 f_4 - f_1 f_2]. \tag{5}$$

The symmetric transition probability w_{1234} includes the constraint

$$\Delta = \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\delta[h(E_1, E_2) - h(E_3, E_4)]. \quad (6)$$

In equilibrium the distributions depend on the phase space points through the energy variables only and the detailed balance principle requires

$$f(E_1)f(E_2) = f(E_3)f(E_4). \quad (7)$$

With the generalized constraint (6) this relation is satisfied by

$$f(E) = f(0)\exp[-X(E)/T] \quad (8)$$

with $1/T = -f'(0)/f(0)$ and $X(E)$ given by Eq. (3). In the extensive case $h(x, y) = x + y$ leads to $X(E) = E$; for the Tsallis-type energy addition rule [19,20],

$$h(x, y) = x + y + axy, \quad (9)$$

one obtains $X(E) = \frac{1}{a}\ln(1 + aE)$ and

$$f(E) = f(0)(1 + aE)^{-1/aT}. \quad (10)$$

Since the energy addition rule (1) conserves the quantity $h(E_1, E_2)$ in a microcollision, the new energies after the collision also lie on the $h =$ constant line. Because of the additivity of the quasienergy, $X(E)$, the total sum, $X_{\text{tot}} = \sum_i X(E_i)$, is a conserved quantity.

Figure 1 presents results of a simple test particle simulation with the rule (9). We mostly started with a uniform energy distribution between zero and $E_0 = 1$ with a fixed number of particles $N = 10^4$ (red, full line). The one-particle energy distribution evolves towards the well-known exponential curve for $a = 0$, shown in the left part of Fig. 1. This snapshot was taken after 200 two-body collisions per particle (blue, short dashed line). The analytical fit to this histogram is given by $2e^{-E/T}$ (with $T = E_0/2 = 0.5$ in this case). An intermediate stage of the evolution after 0.4 collisions per particle is also plotted in this figure (green, long dashed line). Using the prescription with $a = 1$, the stationary solution becomes a Tsallis distribution. The numerical final state from the uniform energy distribution can be inspected in the right side of Fig. 1.

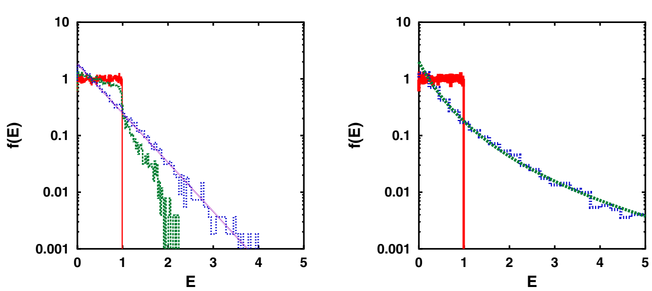

FIG. 1 (color online). Evolution towards the Boltzmann distribution for $a = 0$ (left part) and towards the Tsallis one for $a = 1$ (right part) using $h(E_1, E_2) = E_1 + E_2 + aE_1E_2$.

The fit to this curve is given by $f(E) = 2.6(1 + aE)^{-3.6}$. All distributions are normalized to one.

It is in order to make some remark on the energy conservation. For $h(x, y) = x + y$ we simulate a closed system with elastic collisions: the sum, $U = \sum_{i=1}^N E_i$, does not change in any of the binary collisions. The situation changes by using a nonextensive formula for $h(x, y)$. With a constant positive (negative) a, the bare energy sum, U, is decreasing (increasing) while approaching the stationary distribution. This is typical for open systems gaining or losing energy during their evolution towards a stationary state.

One may be inclined to consider the conserved quasienergy, $X(E)$, as an in-medium one-particle energy. The interesting point is that in general any prescription, $h(E_1, E_2)$—defining a version of the nonextensive thermodynamics—is equivalent to considering a quasienergy, $X(E)$. For small energies one expects a restoration of the extensive rule and $X(E) \approx E$, $X'(0) = 1$. Whenever the pair energy is repulsive (attractive), $h(E_1, E_2) \geq E_1 + E_2$ $[h(E_1, E_2) \leq E_1 + E_2]$, a rising quasienergy is smaller (bigger) than the free one, $X(E) \leq E$ $(X(E) \geq E)$. This leads to a tail of the stationary distribution in the free single-particle energy, $f(E)$, which is above (below) the exponential curve. This phenomenon is hard to distinguish from a power-law tail numerically.

The question arises that—constrained by the conserved number of particles, $N = \int f d\Gamma$, and the total quasienergy, $X_{\text{tot}} = \int f X(E) d\Gamma$—what is the proper formula for the entropy which grows when approaching the stationary distribution. If the addition rule of the nonextensive entropy, s, is given by $h_s(x, y)$, then the quasientropy, $X_s(s)$, is additive, too, and the total entropy is given by $X_s(S_{\text{tot}}) = \int f X_s[s(f)] d\Gamma = \int \sigma(f) d\Gamma$. Its rate of change, $\dot{X}_s(S_{\text{tot}}) = \int \dot{f} \sigma'(f) d\Gamma$ can be expressed with the help of the Boltzmann equation (5). Assuming the symmetry properties $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and $(12) \leftrightarrow (34)$ for the constrained rate factor $w_{1234} = \mathcal{W}\Delta$, one easily derives

$$X'_s(S_{\text{tot}})\dot{S}_{\text{tot}} = \frac{1}{4}\int_{1234} w_{1234}(f_3 f_4 - f_1 f_2)[\sigma'(f_1) + \sigma'(f_2) - \sigma'(f_3) - \sigma'(f_4)]. \quad (11)$$

A definite sign for this quantity can be obtained due to the additivity of $X_s[s(f)] = \sigma(f)/f$, leading to the unique solution $X_s[s(f)] = B\ln f$. With $B = -k_B$ (Boltzmann's constant) $\dot{S}_{\text{tot}} \geq 0$ follows. Using the $k_B = 1$ unit system we arrive at $X_s[s(f)] = -\ln f$, and the expression for the total additive quasientropy coincides with Boltzmann's original suggestion. At the same time, applying a nonextensive addition rule for the entropy, $h_s(x, y) = x + y + (1 - q)xy$, as Tsallis did, we have Abe's formula [26]: $X_s(s) = \frac{1}{1-q}\ln[1 + (1 - q)s]$, and from $X_s(s) = -\ln f$ one obtains Tsallis' entropy formula $S_T = \int f s(f) = \int \frac{f^q - f}{1-q}$. The problem with this formula is that it expresses an integral of a nonadditive entropy density. In numerical simulations we nevertheless observe both entropy integrals and find them to evolve parallel to a high degree (cf. Fig. 2). Asides some binning fluctuations both expressions level off eventually. The reason may lie in the closeness of $X_s(t)$ to the identity for the parameter range applied here.

Physical realizations of nonextensive systems may be discovered depending on our knowledge about the microscopical forces influencing the particles during the pair interactions. For such forces being repulsive, the canonical one-particle energy distribution has a tail above the exponential curve for attractive interactions below. The Tsallis distribution, irrespective from which entropy expression it has been derived in a static theory, fits multiplicity distributions as well as p_T spectra with power-law tail in elementary particle reactions [27].

In the quark-gluon plasma (QGP), or more generally in a parton matter before hadronization, color nonsinglet objects are the single particles. Eventually all form hadrons in the soft sector perhaps by recombination and in the hard sector dominantly by fragmentation. In both cases a long-range interaction between color nonsinglet partons, connected to the physical phenomenon confinement, is present in the background. In the following we consider a simple, speculative model for including this type of nonperturbative pair interaction.

For the sake of simplicity let us restrict ourselves to two-body processes between color triplets and antitriplets. This is the most common way of meson formation. It is also an important part of baryon formation due to quark-diquark fusion. The pairs of such partons, while they constantly interact, are either in a color singlet or in a color octet state The energy of the two-parton system is given by $E_{12}^{\text{color state}} = E_1 + E_2 + \Delta^{\text{color state}}$, where the color average is supposed to be vanishing, $\Delta^{\text{singlet}} + 8\Delta^{\text{octet}} = 0$. This is certainly the case for interactions like in the Heisenberg model of magnets, where the pair potential is proportional to the product of symmetry generators in the corresponding spin representation. For SU(3) color this is also the case.

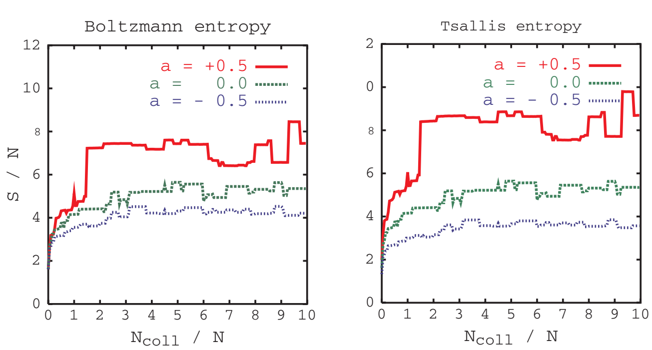

FIG. 2 (color online). Evolution of the Boltzmann and Tsallis entropies by applying the energy addition rule $h(E_1, E_2) = E_1 + E_2 + aE_1E_2$ with different values of the parameter a.

For considering the possibility of a non-Boltzmann distribution in quark matter we further assume a Coulomb-like interaction. In this case $\Delta^{\text{singlet}} = 2E_{12}^{\text{rel.kin.}}$ from the binding in the color singlet channel. For the search after a stationary single-quark distribution of the two-body Boltzmann equation in the octet channel it accounts to consider, $E_{12}^{\text{octet}} = E_1 + E_2 + E_{12}^{\text{rel.kin.}}/4$. The rest is kinematical consideration. We assume the coalescence of two massless partons to a (nearly) massless hadron. Because of the triangle inequality, the kinetic energy of the relative motion of two massless partons is non-negative, $E_{12}^{\text{rel.kin.}} = |\vec{p}_1| + |\vec{p}_2| - |\vec{p}_1 + \vec{p}_2| \geq 0$. For small relative angles between the momentum vectors, ϑ, this is approximated by

$$E_{12}^{\text{rel.kin.}} = \frac{2E_1E_2\sin^2(\vartheta/2)}{E_1 + E_2}. \tag{12}$$

The sum of the individual parton energies in the same approximation is close to $E_1 + E_2 \approx P = |\vec{p}_1 + \vec{p}_2|$. Even very hard partons with a high value of the total pair momentum, P, need a little relative motion for interacting: in the singlet channel to eventually form hadrons, in the octet channel to maintain a single-particle quark distribution typical for the prehadronic phase. The stationary version of this distribution, while detailed balance is satisfied on the two-body level, is often found to be close to the Tsallis distribution. We propose that the above mechanism, leading to $E_c = 1/a = 2P/\sin^2(\vartheta/2)$, may be in the background of such findings. Asymptotic freedom is recovered as for very fast partons $E_c \to \infty$ with $P \to \infty$, and so the one-particle energies of a colliding pair become additive.

In conclusion, we have investigated deterministic, nonextensive energy addition rules in two-body collisions. We have pointed out that instead of the one-particle energy a quasienergy is conserved by such rules in each collision, leading to a non-Boltzmannian stationary distribution in the bare one-particle energy. In particular, the Tsallis distribution is obtained by using a Tsallis-type nonextensive energy addition rule. The Boltzmann entropy, $S_B = X_s(S_{\text{tot}}) = -\int f \ln f d\Gamma$, is never decreasing and reaches its maximum at this distribution. Alternative expressions for the entropy, in particular, the one promoted by Tsallis, correspond to a nonextensive entropy addition rule which defines $X_s(s)$. The Tsallis-type energy and entropy addition rules are related by $q = 1 - aT$.

As a possible physical realization we have proposed a mechanism leading to nearly Tsallis-distributed quarks in quark matter and hadrons which eventually form. This mechanism considers a color state dependent pair energy. The use of a virial theorem connects the color interaction with kinematical factors of the quark pair. In a certain approximation the modification of the familiar two-body energy conservation factor in the Boltzmann equation receives a term proportional to the product of single-quark kinetic energies to leading order in the ultrarelativistic expansion. This leads to a power-law tailed energy distribution. We note that the present approach, similar to the superstatistics [28], can be applied to distributions more general than the Tsallis one. The non-Gaussian behavior of large subsystems and their fluctuations may be reflected in the fact that the conserved energy, $E_{\rm tot} = X^{-1}[\sum_i X(E_i)]$, contains up to N-particle product terms.

Our mathematical model can be used either (i) to describe equilibrium states with exponential distribution in a quasienergy with a nontrivial dispersion relation, $X[E(|\vec{p}|)]$, typical for medium time-scale long-range interactions, or (ii) as an effective algorithm to simulate metastable or stationary open systems with particular energy distributions, $f(E)$, by using simple collision rules. In some real physical systems the energy composition rule may relax to the additive one on the long term; in some other situations, like hadronization of quark matter, the detected particles already follow a free dispersion relation, but their distribution belongs to an earlier, highly correlated state.

Enlightening discussions with Professor Carsten Greiner and Dr. Zhe Xu at the University of Frankfurt as well as with Dr. Géza Györgyi at Eötvös University and Dr. Antal Jakovác at the Technical University Budapest are gratefully acknowledged. This work has been supported by the Hungarian National Science Fund OTKA (T34269, T49466) and by the DFG.

*On leave from RMKI Budapest, Hungary.

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