

Incompressible Liquid State of Rapidly Rotating Bosons at Filling Factor 3/2

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Bosons in the lowest Landau level, such as rapidly rotating cold trapped atoms, are investigated numerically in the specially interesting case in which the filling factor (ratio of particle number to vortex number) is 3/2. When a moderate amount of a longer-range (e.g., dipolar) interaction is included, we find clear evidence that the ground state is in a phase constructed earlier by two of us, in which excitations possess non-Abelian statistics.

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There is increasing interest in rapidly rotating ultracold atoms in a trap. For bosons rotating at a moderate frequency, a vortex lattice is observed [1]. At sufficiently high rotation frequency (close to the natural frequency of a harmonic trap), it is expected that the lattice melts and is replaced by a series of highly correlated liquids that can be related to fractional quantum Hall (QH) states. The crucial parameter is the ratio $\nu = N/N_V$ of the number of atoms N to the number N_V of quantized vortices that would pierce the cloud if it were a Bose condensate. This ratio corresponds to the Landau level filling factor ν in the fractional quantum Hall effect. Previous theoretical work on this regime has emphasized the importance of the lowest Landau level (LLL) when interactions are weak and the temperature is low, and pointed out that in this restricted space of states the Laughlin $\nu = 1/2$ state [2] is the exact ground state for the standard “contact” form of interaction [3]. Later, evidence of a sequence of correlated liquids was found at $\nu = k/2$, $k = 1, 2, 3, \dots$, for $\nu \leq \nu_c$ [4], and a vortex lattice at $\nu > \nu_c$, with $\nu_c \simeq 6-10$ [4,5]. The correlated liquids were found [4] to have large overlaps with states proposed by two of us (RR) some time ago [6]. The $k = 1$ liquid is the familiar Laughlin state, while $k = 2$ is the Moore-Read (MR) paired state [7]. More recent work strengthens the case for the MR state at $\nu = 1$, and provides some evidence for liquid states at still other filling factors not in the sequence $\nu = k/2$ [8]; however, results at $\nu = 3/2$ were inconclusive. Meanwhile, experiments are approaching the LLL regime [9], though the filling factors are still $\gg \nu_c$. Very recently, Bose condensation has been achieved in atoms with a large magnetic dipole moment [10], and the effect of dipolar interactions on the vortex lattices and on the quantum fluids at low filling factors has been investigated [11].

In this Letter, we study numerically the next member, $\nu = 3/2$, of the RR sequence for system sizes larger than in Refs. [4,6,8]. We consider the s -wave (contact) interaction in the LLL and the effect of adding a longer-range component such as the dipolar interaction (for dipole moments oriented parallel to the rotation axis). This case, $k =$

3, is of interest for several reasons: For $k > 1$, the quasi-particle excitations of each of the sequence of states introduced in Refs. [6,7] have the fascinating property of non-Abelian statistics [7]. This makes these states even more exotic than the Laughlin [2] and hierarchy-composite-fermion [12] states, in all of which the quasiparticles have fractional, but Abelian, statistics. $k = 3$ is the smallest k value in the RR sequence for which the non-Abelian statistics support universal quantum computation [13]. For this filling factor, $\nu = 3/2$, there is also an alternative candidate, which is a hierarchy-composite-fermion phase. Using both the sphere and the torus geometries (to avoid edge effects), we find clear evidence that the boson system at $\nu = 3/2$ with a moderate amount of longer-range interaction added is in the phase described by our trial states, and hence is non-Abelian; this may also be the case for the pure s -wave interaction, as suggested by Ref. [4]. The evidence comes from the energy spectrum on the torus, which shows a ground-state doublet with very large overlaps with the RR trial states, and a relatively large gap to higher excited states, and from the two-particle correlation function.

The conventional effective interaction Hamiltonian for the atoms, representing s -wave scattering at low momentum, is

$$H_s = g \sum_{1 \leq i < j \leq N} \delta^3(\mathbf{r}_i - \mathbf{r}_j), \quad (1)$$

where $g = 4\pi\hbar^2 a/M$ (a is the s -wave scattering length, and M is the mass of an atom). It is of interest to consider also electric or magnetic dipole interactions between the atoms, of the form

$$H_{\text{dip}} = C_d \sum_{1 \leq i < j \leq N} \frac{\mathbf{p}_i \cdot \mathbf{p}_j - 3(\mathbf{n}_{ij} \cdot \mathbf{p}_i)(\mathbf{n}_{ij} \cdot \mathbf{p}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad (2)$$

where $\mathbf{n}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|$, and the \mathbf{p}_i 's are *unit* vectors representing the (fixed) dipole moments. We will assume that the dipole moments are parallel to the 3 axis; then any δ -function term that may accompany the dipolar

interaction can be absorbed into the s -wave interaction term in $H_{\text{int}} = H_s + H_{\text{dip}}$.

We work in an axially symmetric harmonic trap, with frequencies ω_3 , ω_{\perp} for motion, respectively, along, and perpendicular to, the symmetry (3) axis. For a system of bosons rotating rapidly about the 3 axis, we will restrict the atoms to the LLL states, which are the single-particle states of lowest energy for each value $m \geq 0$ of angular momentum L_3/\hbar (and thus have no excitation along the 3 axis). This is based on the assumption that interactions are weak; that is, $g\bar{n}$ and $C_d\bar{n}$ (\bar{n} is the typical density of particles in the drop of atoms) are small compared with the energy for excitation to higher states, that is, with $2\hbar\omega_{\perp}$ and $\hbar\omega_3$ [3,4]. The LLL states can be represented by functions in two dimensions, $u_m(z) = z^m e^{-|z|^2/4}/\sqrt{2\pi 2^m m!}$, where $z = x + iy$ (henceforth, we use units in which the magnetic length ℓ_B and \hbar are 1 [14]).

When working on the sphere [15], there is a finite number N_V of flux quanta penetrating the surface of radius $R = \sqrt{N_V/2}$, and there are $N_V + 1$ single-particle states, which have wave functions $u_m(z) \propto z^m/(1 + |z|^2/4R^2)^{1+N_V/2}$, $m = 0, \dots, N_V$ in terms of the coordinate z in the plane (from stereographic projection of the sphere). In the torus geometry (i.e., periodic boundary conditions on a parallelogram), there are N_V LLL basis states when N_V flux pierce the system.

In the subspace of LLL states in the infinite plane, the interaction Hamiltonian H_{int} can be represented by the pseudopotentials V_m , $m = 0, 2, \dots$. V_m is the interaction energy for a single pair of bosons of relative angular momentum m (only even m are relevant for bosons) [15]. The pseudopotentials can be obtained [11] from the analytic expressions in the limit (for simplicity) of vanishing thickness of the two-dimensional fluid, $\ell_3/\ell_{\perp} \rightarrow 0$ [14]. In this limit an infinite constant has to be absorbed into g . The $m \neq 0$ pseudopotentials are determined entirely by the dipolar interaction, and hence their ratios are fixed; they are plotted as an inset in Fig. 2 below. V_0 can be treated as an independent parameter, so that the ratio of V_2/V_0 is a dimensionless parameter characterizing the interaction apart from one overall energy scale. We also studied an interaction in which V_m for $m > 2$ is set to zero. This description of the interaction can be extended to the sphere (also using rotation symmetry) and to the torus.

A p -body δ -function interaction,

$$H_p = \frac{W_p}{p!} \sum_{i_1, i_2, \dots, i_p=1}^N \delta^2(z_{i_1} - z_{i_2}) \cdots \delta^2(z_{i_{p-1}} - z_{i_p}), \quad (3)$$

projected to the LLL is also of interest (this form is correct for the plane and torus geometries, and for the sphere there is a corresponding rotationally invariant form). The parafermion states found in Ref. [6] are unique, exact zero-energy eigenstates of such interactions for $p = k + 1$ when N is divisible by k and $N_V = 2N/k - 2$ (on the

sphere), so that $\nu = \lim_{N \rightarrow \infty} N/N_V = k/2$. These states serve as trial wave functions with which the exact ground states for H_{int} can be compared. They represent incompressible liquid phases, in which the excitations enjoy non-Abelian statistics for $k > 1$ (however, we will later suggest that a caveat to this statement is required).

Before turning to our results, we review some aspects of the LLL on a torus [16]. On the torus at $\nu = N/N_V = 3/2$, translational symmetry implies that all energy eigenstates possess a trivial center-of-mass degeneracy of 2, which is exact for any size system, and also that there is a conserved pseudomomentum \mathbf{K} [16], which is a vector lying in a certain Brillouin zone. In the RR phases, the ground states have a net degeneracy $k + 1$ in the thermodynamic limit, which is connected with the non-Abelian statistics [6,7]. For $k = 3$, this fourfold degeneracy is made up of the trivial factor 2 (which is always discarded in numerical studies), together with a further twofold degeneracy, which in general becomes exact only in the thermodynamic limit; all these ground states have $\mathbf{K} = \mathbf{0}$. (For the 4-body interaction, the fourfold degeneracy is exact for any size, as the ground states have exactly zero energy.) By contrast, the hierarchy-composite-fermion ground state for bosons at $\nu = 3/2$ possesses only the trivial twofold degeneracy. Then for an incompressible fluid on the torus, the spectrum of a sufficiently large system in one of these two phases should exhibit a nearly degenerate pair of ground states (respectively, a single ground state) at $\mathbf{K} = \mathbf{0}$, separated by a clear gap from a region of many states at higher energy eigenvalues.

In Fig. 1 we show the spectrum for 18 particles on the torus for H_{int} with $V_2/V_0 = 0.380$ (energies are in units of C_d for the dipolar H). A clear doublet can be seen at $\mathbf{K} = \mathbf{0}$, and the rest of the spectrum is well-separated compared with either the spacing of the doublet or the spacing among the levels above the gap. This spectrum leaves little doubt that the fluid is incompressible. The results are similar for the two geometries of the torus (hexagonal and square unit

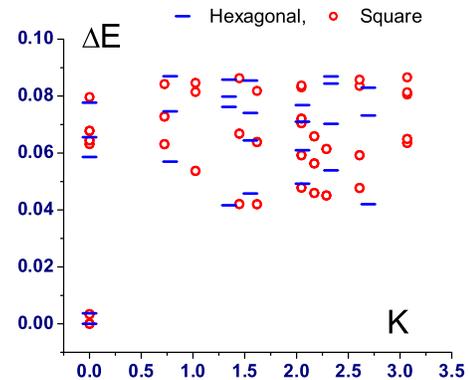


FIG. 1 (color online). Low-lying spectrum for 18 bosons for dipolar model H_{int} with $V_2/V_0 = 0.380$ on the torus vs the pseudomomentum $K = |\mathbf{K}|$, for two different unit cells.

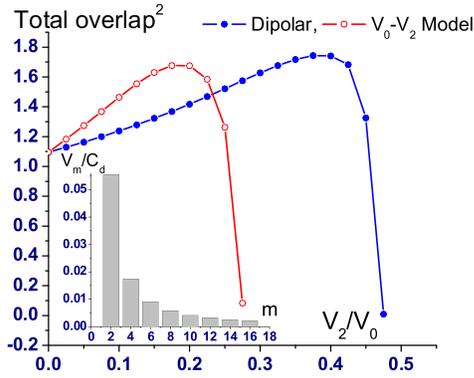


FIG. 2 (color online). The total overlap squared (sum of the two RR wave functions with the lowest two $K = 0$ states of H_{int}) as a function of V_2/V_0 on the torus (square unit cell), for 18 particles, for both the dipolar and the V_0 - V_2 -only model interactions. The inset shows the $m \neq 0$ pseudopotentials for the dipolar interaction.

cells) shown. Spectra at nearby V_2/V_0 , and at other aspect ratios, are similar.

The overlap of each of the two low-lying $\mathbf{K} = \mathbf{0}$ states with the trial ground-state doublet of the 4-body interaction can be calculated. We may add the squares of these overlaps to obtain the total overlap squared (which is at most 2) of the two-dimensional subspace of H_{int} with that of H_4 . This total overlap squared is plotted vs V_2/V_0 for both the V_0 plus dipole interaction and the V_0 - V_2 -only interaction in Fig. 2 (the total is made up of roughly equal contributions from each of the two low-lying states, across the whole range of V_2/V_0 values). For sufficiently large V_2/V_0 , both cases show a very abrupt drop in overlap which is due to a phase transition. Beyond this point, the low-lying spectrum shows indications that the ground state in the thermodynamic limit breaks translational symmetry. The transition is probably first order, but does not occur here by crossing of energy levels of different symmetry, because a crystal and the RR fluid both have ground states at $\mathbf{K} = \mathbf{0}$. This transition is similar to that found in other recent work at $\nu = 1/2$ [11] and will not be pursued here. We note, however, the similarity of these overlap curves with those for $\nu = 1/2$ in the case of fermions as the interaction is changed, which show similar trends and similar curves [17].

The $N = 18$ overlaps should be viewed as significantly large. Note that for a random vector in D dimensions, the probability of obtaining an overlap with a given vector (or two-dimensional subspace) greater than some given value is of order e^{-D} as $D \rightarrow \infty$. The probability of obtaining similar overlaps for two random vectors is of order the square of this. In our case, the $\mathbf{K} = \mathbf{0}$ block of the matrix contains (with reflection symmetry included) about 242 000 states. However, there are other symmetries that we did not use, such as point operations, that commute with translations at the zone center and would further reduce

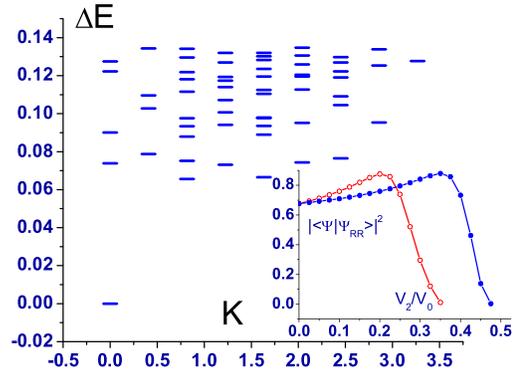


FIG. 3 (color online). Low-lying spectrum for 21 particles on the sphere for dipolar interaction with $V_2/V_0 = 0.350$ vs equivalent wave vector $K = L/R$ (L is the angular momentum). Inset shows the overlap squared, with symbols as in Fig. 2.

somewhat the dimension of the relevant space. The large value of not only the ground state, but also the lowest excited state, overlap with the trial subspace is very strong evidence that these systems are in the RR phase, and would not support an interpretation as hierarchy-composite-fermion states.

Figure 3 shows the spectrum in the spherical geometry, with $V_2/V_0 = 0.350$. Again, a clear gap separates the ground state from the rest of the spectrum. The results are consistent with those on the torus.

The two-particle correlation function $g(r)$ for the ground state on the sphere is shown in Fig. 4, for the V_0 - V_2 -only interaction for 21 particles, together with that of the RR trial state (the ground state of H_4) for 18 particles, for comparison. In general, $g(r)$ in a strongly correlated state should show a correlation hole at short distances [i.e., $g(0) < 1$], and in an incompressible fluid state it should tend to 1 exponentially in r at long distances. In the present case, while $g(r)$ is less than 1 at $r = 0$, it has a local maximum there (mentioned in Ref. [8], and this seems to be a general feature of the ground states for $\nu > 1/2$). This surprising result may indicate a general tendency for bo-

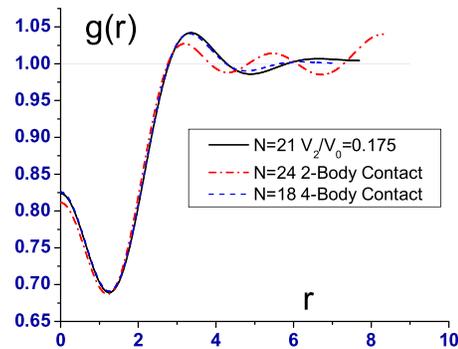


FIG. 4 (color online). Two-particle correlation function $g(r)$ of various ground states on the sphere, plotted against great-circle distance r . $N = 21$ $g(r)$ is for a V_0 - V_2 model.

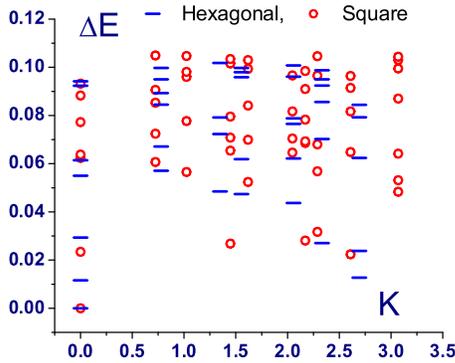


FIG. 5 (color online). Low-lying spectrum (in units of g) for 18 particles on the torus vs pseudomomentum K , for H_s for two different unit cells.

sons to cluster, at least in these rotationally invariant ground states. The behavior at large r suggests that these systems are on the verge of being larger than the relevant correlation length. For the RR trial state, $g(r)$ for 18 particles has essentially converged to its thermodynamic limit.

We can gain insight into the structure of $g(r)$ as ν increases by considering the vortex lattice. In this state there are holes in the density at some fixed positions. Accordingly, $g(r)$ in the vortex lattice, averaged over shifts and rotations of the lattice to represent the finite-size system, will have maxima and minima at arbitrarily large r ; in particular, there will be a maximum at $r = 0$. It seems that, in the RR series, the oscillations in $g(r)$ die out with increasing r at least for small k , but that as k increases the oscillations extend out to larger r , and the maximum at $r = 0$ becomes larger. We suspect that for sufficiently large k , the RR trial states actually exhibit a transition to vortex-lattice long-range order (or possibly to one of the other ordered states found in Ref. [11]). This would be analogous to the Laughlin states at $\nu = 1/m$, which exhibit increasing oscillations with increasing m , and eventually develop crystalline order of the particles [$g(0) = 0$ in these cases] [2]. Note that in both cases the long-range order in the trial states is (or would be) occurring for the exact ground states of some family of special Hamiltonians, which are not the ones of most physical interest, yet the transition is of the same type as that for the latter (though the critical ν may be much different, as for the Laughlin states).

Figure 5 shows the spectrum of the pure contact interaction H_s (i.e., $V_2/V_0 = 0$) on the torus. This, in conjunction with the corresponding $g(r)$ on the sphere for $N = 24$ particles shown in Fig. 4, shows no clear signal that the system is incompressible. While it is possible that the correlation length is simply larger, and that larger systems would exhibit incompressibility, we cannot rule out the

possibility that the fluid in this region is compressible or that there is a breaking of translational symmetry in the thermodynamic limit. Indeed, the dependence of the spectrum on the aspect ratio in the torus geometry suggests the presence of a broken symmetry “stripe” phase competing with the RR phase [18].

In conclusion, we have exhibited clear evidence that for bosons in the lowest Landau level at filling factor $3/2$, when a moderate amount of longer-range interaction is included, the ground state is an incompressible fluid of a type that possesses non-Abelian statistics for the quasiparticle excitations.

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