Cosmological Solutions in Macroscopic Gravity

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In the macroscopic gravity approach to the averaging problem in cosmology, the Einstein field equations on cosmological scales are modified by appropriate gravitational correlation terms. We present exact cosmological solutions to the equations of macroscopic gravity for a spatially homogeneous and isotropic macroscopic space-time and find that the correlation tensor is of the form of a spatial curvature term. We briefly discuss the physical consequences of these results.

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The averaging problem in cosmology and general relativity (GR) is of fundamental importance [1]. An averaging of inhomogeneous space-times can lead to dynamical behavior different from the spatially homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) model; in particular, the expansion rate may be significantly affected [2]. This motivated the macroscopic gravity (MG) approach to the averaging problem in cosmology, in which the Einstein equations on the cosmological scales with a continuous distribution of cosmological matter are modified by appropriate gravitational correlation correction terms [3].

There are a number of approaches to the averaging problem [2,4]. The perturbative approach involves averaging the perturbed Einstein equations; however, a perturbation analysis cannot provide any information about an averaged geometry. In the space-time or space volume averaging approach tensors, and in some cases only scalar quantities, are averaged; this procedure is not generally covariant, and hence the results are somewhat limited and the conclusions unreliable. In all of these approaches, in analogy with Lorentz's approach to electrodynamics, an averaging of the Einstein equations is performed to obtain the averaged field equations. But to date, with the exception of the MG approach [3], no proposal has been made about the correlation functions which should inevitably emerge in an averaging of a nonlinear theory (without which the averaging of the Einstein equations simply amount to definitions of the new averaged terms).

In particular, approaches to describe FLRW cosmologies as locally inhomogeneous cosmological models utilize a $3+1$ cosmological space-time splitting with noncovariant space volume averaging. The size of the averaging space regions has been tacitly assumed to be $\simeq 100$ Mpc, or of the order of the inverse Hubble scale. Athough many of the approaches have indicated that Friedmann's equation gets modified by the appearance of an effective averaged energy density, there is no definite consensus as of yet on the physical status and mathematical reliability of this important prediction on the possible dynamical law of the Universe evolution on its largest scales.

The space-time averaging procedure adopted in MG is based on the concept of Lie dragging of averaging regions, which makes it valid for any differentiable manifold with a volume n form, and it has been proven to exist on arbitrary Riemannian space-times with well-defined local averaged properties [3]. Averaging of the structure equations for the geometry of GR brings about the structure equations for the averaged (macroscopic) geometry and the definitions and the properties of the correlation tensors. The averaged Einstein equations for the macroscopic metric tensor together with a set of algebraic and differential equations for the correlation tensors become a coupled system of the macroscopic field equations for the unknown macroscopic metric, correlation tensor, and other objects of the theory. The averaged Einstein equations can always be written in the form of the Einstein equations for the macroscopic metric tensor when the correlation terms are moved to the right-hand side of the averaged Einstein equations to serve as the geometric modification to the averaged (macroscopic) matter energy-momentum tensor. Thus, MG is a geometric field theory with a built-in scale which is nonperturbative and provides us with both the geometry underlying the macroscopic gravitational phenomena and the macroscopic (averaged) field equations [3]. The scale is given by the size of the space-time averaging regions which is a free parameter of the theory. When applied to study cosmological evolution, the theory of MG can be regarded as a long-distance modification of GR.

A procedure for solving the system of MG equations with one connection correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma}$ (in brief $\mathbf{Z}$) is as follows. [The MG equations are described in detail in [3,5] (wherein all terms are defined); in order to make this Letter as easy to read as possible, we shall simply present the necessary details in a brief and compact fashion.] The line element for the macroscopic geometry is given in terms of the macroscopic metric tensor $G_{\alpha\beta}$; its Levi-Civita connection coefficients and the Riemannian curvature tensor $M^{\alpha}{}_{\beta\gamma\delta}$ can be calculated in terms of the unknown metric functions. The components of $\mathbf{Z}$, perhaps with an assumption on their functional form based on symmetries and physical conditions, can then be expressed

in terms of the metric functions. The integrability conditions for the differential equations (the **ZM** equations)

$$Z^{\epsilon}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma}M^{\alpha}{}_{\underline{\epsilon}\lambda\rho]} - Z^{\alpha}{}_{\epsilon[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma}M^{\epsilon}{}_{\underline{\beta}\lambda\rho]} + Z^{\alpha}{}_{\beta[\gamma}{}^{\epsilon}{}_{\underline{\nu}\sigma}M^{\mu}{}_{\underline{\epsilon}\lambda\rho]} - Z^{\alpha}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\epsilon}\sigma}M^{\epsilon}{}_{\underline{\nu}\lambda\rho]} = 0, \quad (1)$$

where an underbar denotes that index is not included in the antisymmetrization, are solved. The system of differential equations for **Z** (the **dZ** equations)

$$Z^{\alpha}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma\|\lambda]} = 0, \quad (2)$$

where "$\|$" denotes covariant differentiation with respect to the macroscopic metric, are then solved. Finally, the quadratic algebraic conditions for $Z^{\alpha}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma]}$ (the **ZZ** equations)

$$Z^{\delta}{}_{\beta[\gamma}{}^{\theta}{}_{\underline{\kappa}\pi}Z^{\alpha}{}_{\underline{\delta}\epsilon}{}^{\mu}{}_{\underline{\nu}\sigma]} + Z^{\delta}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma}Z^{\theta}{}_{\underline{\kappa}\pi}{}^{\alpha}{}_{\underline{\delta}\epsilon]} + Z^{\alpha}{}_{\beta[\gamma}{}^{\delta}{}_{\underline{\nu}\sigma}Z^{\mu}{}_{\underline{\delta}\epsilon}{}^{\theta}{}_{\underline{\kappa}\pi]} + Z^{\delta}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\delta}\epsilon}Z^{\theta}{}_{\underline{\kappa}\pi}{}^{\delta}{}_{\underline{\nu}\sigma]} + Z^{\alpha}{}_{\beta[\gamma}{}^{\theta}{}_{\underline{\delta}\epsilon}Z^{\mu}{}_{\underline{\nu}\sigma}{}^{\delta}{}_{\underline{\kappa}\pi]} + Z^{\alpha}{}_{\beta[\gamma}{}^{\delta}{}_{\underline{\kappa}\pi}Z^{\theta}{}_{\underline{\delta}\epsilon}{}^{\mu}{}_{\underline{\nu}\sigma]} = 0, \quad (3)$$

are solved. Upon determining the components of **Z**, the gravitational stress-energy tensor $T_{\beta}^{\alpha(\mathrm{grav})}$ of MG, defined by

$$\left(Z^{\alpha}{}_{\mu\nu\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}Q_{\mu\nu}\right)\bar{g}^{\mu\nu} = -\kappa T_{\beta}^{\alpha(\mathrm{grav})}, \quad (4)$$

is determined (where $Z^{\alpha}{}_{\mu\nu\beta} \equiv 2Z^{\alpha}{}_{\mu\epsilon}{}^{\epsilon}{}_{\nu\beta}$, $Z^{\epsilon}{}_{\mu\nu\epsilon} \equiv Q_{\mu\nu}$). The averaged Einstein equations

$$G^{\alpha\epsilon}M_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}G^{\mu\nu}M_{\mu\nu} = -\kappa\langle \mathbf{t}_{\beta}^{\alpha(\mathrm{micro})}\rangle - \kappa T_{\beta}^{\alpha(\mathrm{grav})} \quad (5)$$

are then solved for the unknown metric functions, assuming (for example) that the averaged microscopic stress-energy tensor $\langle \mathbf{t}_{\beta}^{\alpha(\mathrm{micro})}\rangle$ is of a perfect fluid form. [The macroscopic field equations (5) are written in the form of the Einstein equations of GR, with a "modified" stress-energy tensor consisting of the averaged microscopic stress-energy tensor $\langle \mathbf{t}_{\beta}^{\alpha(\mathrm{micro})}\rangle$ and an additional effective stress-energy tensor $T_{\beta}^{\alpha(\mathrm{grav})}$ (4) arising from the correlation tensor **Z** [3,5].]

Given a macroscopic metric $G_{\alpha\beta}$, the calculational procedure is to seek a solution **Z** satisfying the **ZM**, **dZ**, and **ZZ** equations. By making extensive use of GRTENSORII, MAPLE [6], the first step is to define the connection correlation tensor with its rank and symmetries. In practice, a file is created for a rank 6 tensor **Z** possessing no symmetries; the symmetries on **Z** are then imposed by solving systems of algebraic equations. The choice of metric at this stage is irrelevant. Although solving the **ZZ** equations does not involve the metric, we have found it convenient to solve this equation last; since it is quadratic, many solution sets will arise and only after **Z** has been constrained either by **ZM** and **dZ**, or any other additional assumptions on **Z**, is there a possibility of solving **ZZ** computationally. To each solution set of **ZZ** there will be a corresponding $T_{\beta}^{\alpha(\mathrm{grav})}$. A typical worksheet begins with the loading of a macroscopic metric and the connection correlation tensor. It is useful to have a set of the independent components of **Z**. This is easily done by looping through all components of **Z**. We begin by defining a rank 8 tensor corresponding to the **ZM** equations. These algebraic equations are then solved for the independent components of **Z** and the solutions are substituted back into **Z**. It is easily checked that **Z** now satisfies the **ZM** equations. Next we define a rank 7 tensor corresponding to the **dZ** equations. If a solution of these differential equations for the independent components of **Z** can be found, it can then be substituted back into the **Z** tensor. In most of the cases considered, we have found no great computational difficulty in solving the **ZM** and **dZ** equations using MAPLE. At this point the number of independent components of **Z** left unspecified by the **dZ** equations can be computed. To define the **ZZ** equations we define six rank 6 tensors, each corresponding to a term of the **ZZ** equations fully contracted with the Levi-Civita tensor over the antisymmetrized indices. These tensors are calculated individually, then summed to give a rank 6 tensor corresponding to the **ZZ** equations. As above, this tensor can be calculated and its components stored in a set. We then solve for the remaining independent components of **Z**. Since multiple solutions will be obtained, it is necessary to define and calculate multiple copies of the **Z** tensor. There are many variations to the outline given above, depending on the form of the metric and the assumptions on the components of **Z**. For example, assuming that the components of **Z** are all constants in an appropriate form, the **ZM** and **dZ** equations amount to algebraic equations, thus eliminating the need to solve any differential equations. Full details of all of the experiments and techniques are given in [5].

Let us consider a flat spatially homogeneous, isotropic macroscopic FLRW space-time with conformal time η:

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2), \quad (6)$$

where $d\eta = a^{-1}(t)dt$ with a cosmological (coordinate) time t and $a^2(t)$ is an unknown function of the scale factor. It is necessary to make an ansatz for the functional form of the components of **Z** on the basis of symmetries and physical conditions of the macroscopic geometry. The most natural condition on **Z** compatible with the structure of macroscopic space-time (6) is to require all of its components be constant:

$$Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma} = \text{const.} \quad (7)$$

Upon solving the **ZM** and **dZ** equations, using a Maple built-in algebraic system solver and requiring real-valued solutions, we are left with a number of independent com-

ponents in $\mathbf{Z}$ (see [5] for details). Solving the $\mathbf{ZZ}$ equations then yields a number of solutions (with a small number of nonvanishing real-valued components of $\mathbf{Z}$), each of which gives $T_x^{x(\mathrm{grav})} = T_y^{y(\mathrm{grav})} = T_z^{z(\mathrm{grav})} = \frac{1}{3}T_t^{t(\mathrm{grav})}$, where $T_x^{x(\mathrm{grav})} = -\beta/a^2(t)$ and β is a linear combination of the nonzero constant components of $\mathbf{Z}$ (different combinations corresponding to different solutions; e.g., $\beta = -12Z^3{}_{23}{}^3{}_{32}$ in three particular exact solutions with a single independent component of $\mathbf{Z}$). In all cases the MG stress-energy tensor has the form of a perfect fluid with $\rho_c = \beta/\kappa a^2(t)$ and $p_c = -\rho_c/3$ (i.e., $\gamma = 2/3$). After transforming from conformal time η to cosmological time t, we obtain the averaged Einstein equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa\rho}{3} + \frac{\kappa\beta}{3a^2(t)}, \qquad 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\kappa p + \frac{\kappa\beta}{3a^2}. \tag{8}$$

Thus, the averaged Einstein equations for a flat spatially homogeneous, isotropic macroscopic space-time geometry has the form of the Einstein equations of GR for a open spatially homogeneous, isotropic space-time geometry (where the correlation tensor is of the form of a spatial curvature term, with $k = -\beta/3\kappa$). [In principle, without imposing any further conditions, the curvature can be positive or negative. However, if the energy density ρ_c of the MG field is positive, then $\mathrm{tr}(T_\beta^{\alpha(\mathrm{grav})}) = -2\beta/a^2(t) < 0$ (i.e., $\mathbf{T}^{(\mathrm{grav})}$ is a *negative* curvature term), which means from the physical point of view that the macroscopic gravitational energy is the binding energy of the Universe.] In all cases (i.e., calculations in which different assumptions on the form of $\mathbf{Z}$ are made), solutions always give rise to a spatial curvature term. Indeed, assuming only spatial correlations (i.e., assuming that all components of $\mathbf{Z}$ with at least one t index must vanish), it can be shown that $\mathbf{T}^{(\mathrm{grav})}$ must be of the form of spatial curvature [5]. This is the main result of this Letter; namely, *for a flat FLRW geometry the MG correlations are of the form of a spatial curvature tensor term*. In further experimentation, in some nonflat spatially homogeneous, isotropic macroscopic models we also found evidence that $\mathbf{T}^{(\mathrm{grav})}$ is of the form of a curvature term; this will be studied in more detail in [5].

There are a number of important physical consequences of these results. In MG, in a flat spatially homogeneous and isotropic macroscopic space-time, the correlation tensor and the averaged cosmological matter distribution taken as a perfect fluid have the cosmological dynamical equations (8). This implies that the macroscopic (averaged) cosmological evolution in a flat universe is governed by the dynamical evolution equations for an open universe, which makes it necessary to reconsider the standard cosmological interpretation and the treatment of the observational data. If the underlying macroscopic space-time has positive spatial curvature (as suggested by recent observations), then we would obtain a cosmological model which is closed on local scales, but as a result of the MG correlations behaves dynamically on macroscopically large scales as a flat model, which might have considerable physical implications. Finally, if positive spatial curvature correlations are permitted, then cosmological models which act as an Einstein static model on the largest scales are possible even for models with zero or negative curvature on small scales. Thus we have the interesting, but highly conjectural, possibility that since at late times (and on the largest scale) $\mathbf{T}^{(\mathrm{grav})}$ (a curvature term) will dominate the dynamics, the correlations might stabilize the Einstein static model. This may be of potential importance since current observations perhaps indicate that the universe is marginally closed and due to the current interest in the emergent universe scenario in which the universe is positively curved and initially in a past eternal Einstein static state that eventually evolves into a subsequent inflationary phase [7].

Let us discuss the potential significance of these results in a little more detail. Observations are usually interpreted as showing that the Universe is flat, currently accelerating and indicating the existence of dark matter and dark energy [8]. As noted earlier, inhomogeneities can affect the dynamics and may significantly affect the expansion rate [2]. It has been suggested that backreaction from inhomogeneities smaller than the Hubble scale could explain the apparently observed accelerated expansion of the Universe today or negate the need for dark energy in a realistic inhomogeneous universe. Indeed, it has been argued that the cosmological constant can be reduced to a very small value by backreaction effects in an expanding space-time [9]. For example, gravitational waves propagating in a background space-time will affect the dynamics of this background. The backreaction for scalar gravitational perturbations, which can be described by an effective energy-momentum tensor, was studied in [10]. It was found that the equation of state of the dominant infrared contribution to the energy-momentum tensor which describes backreaction can take the form of a negative cosmological constant. This has led to the speculation that gravitational backreaction may lead to a dynamical mechanism for the cancellation of a bare cosmological constant. However, it is not clear whether this approach is consistent and whether the effects are indeed physical. For example, averaging over a fixed time slice, the spatially averaged value of the expansion will not be the same as the expansion rate at the averaged value of time, because of the nonlinear nature of the expansion.

What is needed is a correct averaging procedure that does not depend on any assumptions regarding the nature of the perturbations. The MG method described here is an exact approach; no approximations have been made (i.e., no higher order terms have been dropped). In this approach inhomogeneities affect the dynamics on large scales

through correction terms (and, in this sense, is different to backreaction effects which are pure nonlinear effects of the gravitational field via perturbations). Moreover, averaging entails a scale dependence, which depends on the spatial scale over which averages are taken. This averaging scale is assumed to be of the order of the inverse Hubble scale, and thus any terms (e.g., a cosmological constant or a curvature term) appearing in the correlation tensor must be related to the inverse Hubble scale. For example, the natural length scale of any cosmological constant introduced by averaging would be of the order of the inverse Hubble scale squared. This would therefore give a natural possible resolution of the coincidence problem [9]. Unfortunately, to date we have not been able to solve the MG equations to find a solution with correction terms that may account for the present-day acceleration. However, a spatial curvature correction arises naturally and, as noted earlier, correction terms change the interpretation of observations so that they need to be accounted for carefully to determine if they may be consistent with a decelerating universe.

In addition, superhorizon fluctuations (whose origin is in inflation) affect classical dynamics as measured by local observers (since perturbations affect the expansion rate in a universe with a flat FLRW background). Recently it has been proposed that superhorizon perturbations could explain the present-day accelerated acceleration [11]. However, in [12] it was claimed that the effect proposed in [11] amounts to a simple renormalization of the spatial curvature (essentially a new scale factor can be defined so that the metric looks like a FLRW metric with a curvature term), and thus cannot account for negative deceleration; indeed, a proper accounting of all perturbative terms as well as more general arguments suggest that the superhorizon modes do not lead to acceleration [13]. In further work [14] the relation between backreaction (and especially the effective scale factor presented in [11]) and spatial curvature using exact equations which do not rely on perturbation theory was studied in more detail; it was found that although the effect does not simply reduce to spatial curvature, acceleration results but is accompanied by growth of spatial curvature to an extent that is likely to be incompatible with the cosmic microwave background data.

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