Braid Topologies for Quantum Computation

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In topological quantum computation, quantum information is stored in states which are intrinsically protected from decoherence, and quantum gates are carried out by dragging particlelike excitations (quasiparticles) around one another in two space dimensions. The resulting quasiparticle trajectories define world lines in three-dimensional space-time, and the corresponding quantum gates depend only on the topology of the braids formed by these world lines. We show how to find braids that yield a universal set of quantum gates for qubits encoded using a specific kind of quasiparticle which is particularly promising for experimental realization.

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A quantum computer must be capable of manipulating quantum information while simultaneously protecting it from error and loss of quantum coherence due to coupling to the environment. Topological quantum computation (TQC) [1,2] offers a particularly elegant way to achieve this using quasiparticles which obey non-Abelian statistics [3,4]. These quasiparticles, which are expected to arise in a variety of two-dimensional quantum many-body systems [1,4–11], have the property that the usual phase factors of ± 1 associated with the exchange of identical bosons or fermions are replaced by noncommuting (non-Abelian) matrices that depend only on the topology of the space-time paths (braids) used to effect the exchange. The matrices act on a degenerate Hilbert space whose dimensionality is exponentially large in the number of quasiparticles and whose states have an intrinsic immunity to decoherence because they cannot be distinguished by local measurements, provided the quasiparticles are kept sufficiently far apart.

In TQC this protected Hilbert space is used to store quantum information, and quantum gates are carried out by adiabatically braiding quasiparticles around each other [1,2]. Because the resulting quantum gates depend purely on the topology of the braids, errors occur only when quasiparticles form ''unintentional'' braids. This can happen if a quasiparticle-quasihole pair is thermally created, the pair separates, wanders around other quasiparticles, and then recombines in a topologically nontrivial way. However, such processes are exponentially unlikely at low enough temperature. This built-in protection from error and decoherence is an appealing feature of TQC which may compensate for the extreme technical challenges that will have to be overcome to realize it.

It has been shown that several different kinds of non-Abelian quasiparticles can be used for TQC [1,2,12–14]. Here we focus on what is arguably the simplest of these—Fibonacci anyons [14]. These quasiparticles each possess a ''q-deformed'' spin quantum number (q-spin) of 1, the properties of which are described by a mathematical structure known as a quantum group [15]. As with ordinary spin, there are specific rules for combining q-spin. For Fibonacci anyons these ''fusion'' rules state that when two q-spin 1 objects are combined, the total q-spin can be either 0 or 1; and when a q-spin 0 object is combined with a q-spin s object, where $s = 0$ or 1, the total q-spin is s [16]. Remarkably, as shown in [14], these fusion rules fix the structure of the relevant quantum group, uniquely determining the quantum operations produced by braiding q-spins around one another up to an overall Abelian phase which is irrelevant for TQC.

One reason for focusing on Fibonacci anyons is that they are thought to exist in an experimentally observed fractional quantum Hall state [17,18]. It may also be possible to realize them in rotating Bose condensates [7] and quantum spin systems [10,11]. Strictly speaking, the quantum group realized in some of these systems, and considered for TQC in [2], also includes q-spins of $\frac{1}{2}$ and $\frac{3}{2}$; however, due to a symmetry of this quantum group [6], the braiding properties of q-spin $\frac{1}{2}$ quasiparticles are equivalent to those with q-spin 1, and the braid topologies we find below can be used in either case.

The fusion rules for Fibonacci anyons imply that the Hilbert space of two quasiparticles is two dimensional—with basis states $|(\bullet,\bullet)_0\rangle$ and $|(\bullet,\bullet)_1\rangle$. Here the notation $(\bullet,\bullet)_a$ represents two quasiparticles with total q-spin a. When a third quasiparticle is added, the Hilbert space is three dimensional, and is spanned by the states $|((\bullet,\bullet)_0,\bullet)_1\rangle$, $|((\bullet,\bullet)_1,\bullet)_1\rangle$, and $|((\bullet,\bullet)_1,\bullet)_0\rangle$. The general result is that the dimensionality of an N-quasiparticle state is the $(N+1)$st Fibonacci number. To use this Hilbert space for quantum computation, we follow Freedman *et al.* [2], and encode qubits into triplets of quasiparticles with total q-spin 1, taking the logical qubit states to be $|0_L\rangle = |((\bullet,\bullet)_0,\bullet)_1\rangle$ and $|1_L\rangle = |((\bullet,\bullet)_1,\bullet)_1\rangle$. The remaining state with total q-spin 0 is then a noncomputational state, $|\mathrm{NC}\rangle = |((\bullet, \bullet)_1, \bullet)_0\rangle$. This encoding, illustrated in Fig. 1, can be viewed as a q-deformed version of the three-spin qubit encoding proposed for exchange-only quantum computation [19]. As in that case, qubits can be measured by determining the q-spin of the two leftmost quasiparticles, either by performing local measurements once the quasiparticles are moved close together [1] or possibly by performing interference experiments [20,21]. Similar schemes can be used for initialization. The price for introducing this encoding is that care must now be taken to minimize transitions to noncomputational states, known as leakage errors, when carrying out computations.

Figure 2(a) shows elementary braiding operations for three quasiparticles together with the matrices which describe the transitions they induce in the Hilbert space illustrated in Fig. 1 [2,6,14]. Any three-quasiparticle braid can be constructed out of these elementary operations and their inverses. The corresponding transition matrix can then be computed by simply multiplying the appropriate matrices as shown in Fig. 2(b). The upper 2×2 blocks of these matrices act on the computational qubit space, and the lower right element is a phase which multiplies $|\mathrm{NC}\rangle$. This block diagonal form illustrates that if a group of quasiparticles is in a q-spin eigenstate then braiding of quasiparticles within this group does not lead to transitions out of this eigenstate. It follows that single-qubit gates performed by braiding quasiparticles within a qubit will not lead to leakage error.

To find braids which perform a given single-qubit gate, we first carry out a brute force search of three-quasiparticle braids with up to 46 interchanges. This exhaustive search typically yields braids approximating the desired target gate to within a distance of $\epsilon \sim 1$–2×10^{-3} (here we define distance between gates using the operator norm—see Fig. 3 for a definition). If more accuracy is required, brute force searching becomes exponentially more difficult and rapidly becomes unfeasible. Fortunately, a powerful theorem due to Solovay and Kitaev [22,23] guarantees that given a set of gates generated by finite braids which is sufficiently dense in the space of all gates (easily generated for three quasiparticles) braids approximating arbitrary single-qubit gates to any required accuracy can be found efficiently, with the length of the braid growing as $\sim |\log\epsilon|^c$, where $c \simeq 4$.

We now turn to the significantly more difficult problem of finding braids which approximate a desired two-qubit gate. In this case there are six quasiparticles, and the Hilbert space is 13 dimensional. The elementary braid matrices acting on this space are again block diagonal, with 5×5 (total q-spin 0) and 8×8 (total q-spin 1) blocks [24]. It is known that braiding these six quasiparticles generates a set of unitary operations which is dense in the space of all such block diagonal operations [2,14], and the Solovay-Kitaev theorem again guarantees one can in principle construct braids to approximate any desired operation of this form [22]. However, unlike the single-qubit case, actual implementation of this procedure is problematic. The space of unitary operations for six quasiparticles is parametrized by 87 continuous parameters, as opposed to 3 for the three-quasiparticle case, and searching for braids which approximate a desired quantum gate in this high dimensional space is an extremely difficult numerical problem. To circumvent this difficulty, we have found constructions for a particular class of two-qubit gates (controlled-rotation gates) which require only finding a finite number of three-quasiparticle braids. The resulting reduction of the dimensionality of the search space from 87 to 3 makes it possible for the first time to compile accurate braids for a class of two-qubit gates which can be systematically improved using the Solovay-Kitaev theorem.

Our constructions are based on two essential ideas. First, we *weave* a pair of quasiparticles (the control pair) from one qubit (the control qubit) through the quasiparticles forming the second qubit (the target qubit). By weaving

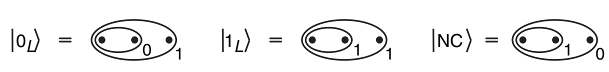

FIG. 1. Basis states for the three-dimensional Hilbert space of three quasiparticles and qubit encoding. The ovals enclose groups of quasiparticles in q-spin eigenstates labeled by the corresponding eigenvalues. The states $|0_L\rangle$ and $|1_L\rangle$ (denoted, respectively, $|((\bullet, \bullet)_0, \bullet)_1\rangle$ and $|((\bullet, \bullet)_1, \bullet)_1\rangle$ in the text) span the computational qubit space, while the state $|\mathrm{NC}\rangle$ (denoted $|((\bullet, \bullet)_1, \bullet)_0\rangle$ in the text) is a noncomputational state.

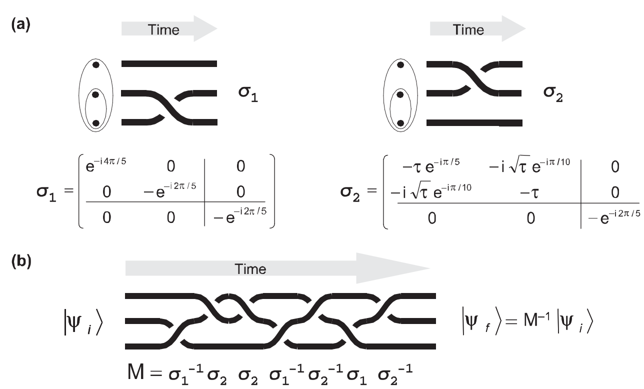

FIG. 2. (a) Elementary three-quasiparticle braids. The pictures represent quasiparticle world lines in $2 + 1$-dimensional spacetime, with time flowing from left to right. The matrices σ_1 and σ_2 are the transition matrices produced by these elementary braids which act on the three-dimensional Hilbert space shown in Fig. 1. Here $\tau = (\sqrt{5} - 1)/2$ is the inverse of the golden mean. The upper 2×2 blocks of these matrices act on the computational qubit space (total q-spin 1) and are used to perform single-qubit rotations, while the lower right element is a phase which multiplies $|\mathrm{NC}\rangle$. (b) A general three-quasiparticle braid and the corresponding matrix expression for the transition matrix it produces. Here $|\psi_i\rangle$ is the initial state and $|\psi_f\rangle$ the final state after braiding. Note that these (and subsequent) figures only represent the topology of the braid. In any actual implementation, quasiparticles will have to be kept sufficiently far apart to keep from lifting the topological degeneracy.

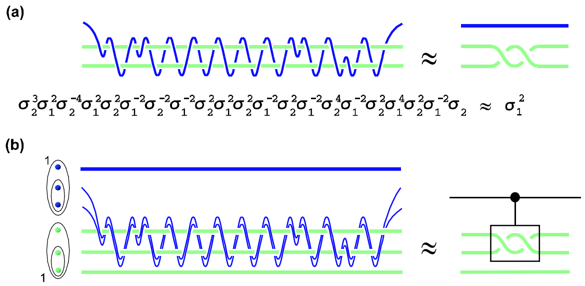

FIG. 3 (color online). (a) A three-quasiparticle braid in which one quasiparticle is woven around two static quasiparticles and returns to its original position (left), and yields approximately the same transition matrix as braiding the two stationary quasiparticles around each other twice (right). The corresponding matrix equation is also shown. To characterize the accuracy of this approximation, we define the distance between two matrices, U and V, to be $\epsilon = \|U - V\|$, where $\|O\|$ is the operator norm of O equal to the square-root of the highest eigenvalue of $O^\dagger O$. The distance between the matrices resulting from the actual braiding (left) and the desired effective braiding (right) is $\epsilon \simeq 2.3 \times 10^{-3}$. (b) A two-qubit braid constructed by weaving a pair of quasiparticles from the control qubit (top) through the target qubit (bottom) using the weaving pattern from (a). The result of this operation is to effectively braid the upper two quasiparticles of the target qubit around each other twice if the control qubit is in the state $|1_L\rangle$, and otherwise do nothing. This is an entangling two-qubit gate which can be used for universal quantum computation. Since all effective braiding takes place within the target qubit, any leakage error is due to the approximate nature of the weave shown in (a). By systematically improving this weave using the Solovay-Kitaev construction, leakage error can be reduced to whatever level is required for a given computation.

we mean that the target quasiparticles remain fixed while the control pair is moved around them as an immutable group [see, for example, Figs. 3(b) and 4(c)]. If the q-spin of the control pair is 0, the result of this operation is the identity. However, if the q-spin of the control pair is 1, a transition is induced. If we choose the control pair to consist of the two quasiparticles whose total q-spin determines the state of the control qubit, this construction automatically yields a controlled (conditional) operation. Second, we deliberately weave the control pair through only two target quasiparticles at a time. Since the only nontrivial case is when the control pair has q-spin 1, and is thus equivalent to a single quasiparticle, this reduces the problem of constructing two-qubit gates to that of finding a finite number of specific three-quasiparticle braids.

Figure 3(a) shows a three-quasiparticle braid in which one quasiparticle is woven through the other two and then returns to its original position. The resulting unitary operation approximates that of simply braiding the two static quasiparticles around each other twice to a distance of $\epsilon \simeq 2.3 \times 10^{-3}$. Similar weaves can be found which approximate any even number, $2m$, of windings of the static quasiparticles. Figure 3(b) shows a two-qubit braid in which the pattern from Fig. 3(a) is used to weave the control pair through the target qubit. If the control qubit is in the state $|0_L\rangle$, this weave does nothing, but if it is in the state $|1_L\rangle$, the effect is equivalent to braiding two quasiparticles within the target qubit. Thus, in the limit $\epsilon \to 0$, this *effective* braiding is all within a qubit and there are no leakage errors. The resulting two-qubit gate is a controlled rotation of the target qubit through an angle of $6m\pi/5$, which, together with single-qubit rotations, provides a universal set of gates for quantum computation provided m is not divisible by 5 [25]. Carrying out one iteration of the Solovay-Kitaev construction [22,23] on this weave using the procedure outlined in [26] reduces ϵ by a factor of ~20 at the expense of a factor of 5 increase in length. Subsequent iterations can be used to achieve any desired accuracy.

A similar construction can be used to carry out arbitrary controlled-rotation gates. Figure 4(a) shows a braid in

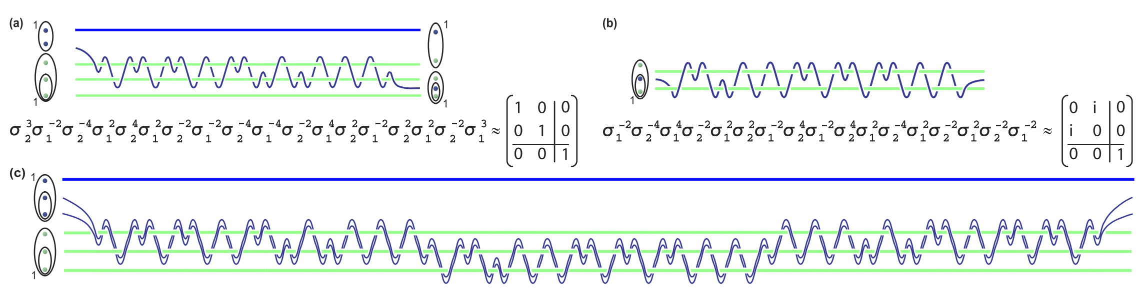

FIG. 4 (color online). (a) An injection weave for which the product of elementary braiding matrices, also shown, approximates the identity to a distance of $\epsilon \simeq 1.5 \times 10^{-3}$. This weave injects a quasiparticle (or any q-spin 1 object) into the target qubit without changing any of the underlying q-spin quantum numbers. (b) A weaving pattern which approximates a NOT gate to a distance of $\epsilon \simeq 8.5 \times 10^{-4}$. (c) A controlled-NOT gate constructed using the weaves shown in (a) and (b) to inject the control pair into the target qubit, perform a NOT operation on the injected target qubit, and then eject the control pair from the target qubit back into the control qubit. The distance between the gate produced by this braid acting on the computational two-qubit space and an exact controlled-NOT gate is $\epsilon \simeq 1.8 \times 10^{-3}$ and $\epsilon \simeq 1.2 \times 10^{-3}$ when the total q-spin of the six quasiparticles is 0 and 1, respectively. Again, the weaves shown in (a) and (b) can be made as accurate as necessary using the Solovay-Kitaev theorem, thereby improving the controlled-NOT gate to any desired accuracy. By replacing the central NOT weave, arbitrary controlled-rotation gates can be constructed using this procedure.

which one quasiparticle is again woven through two static quasiparticles, but this time does not return to its original position. The unitary transformation produced by this weave approximates the identity operation to a distance of $\epsilon \simeq 1.5 \times 10^{-3}$, where, as above, the accuracy of this approximation can be systematically improved by the Solovay-Kitaev theorem. In the limit $\epsilon \rightarrow 0$, the effect of this weave is to permute the three quasiparticles involved without changing any of the underlying q-spin quantum numbers, as shown in the figure. It can therefore be used to safely *inject* a quasiparticle, or any object with q-spin 1, into a qubit. Figure 4(b) then shows a weave which performs an approximate NOT gate on the target qubit. These two weaves are used to construct the two-qubit braid shown in Fig. 4(c). In this braid, the control pair is first injected into the target qubit using the "injection weave." When the control pair has q-spin 1 the state of the modified target qubit is unchanged after injection—the only effect is that one of the target quasiparticles has been replaced by the control pair. A NOT operation is then performed on the injected target qubit by weaving the control pair inside the target using the pattern from Fig. 4(b). In the limit of an exact injection weave this braiding is all within a q-spin eigenstate and there are no leakage errors. Finally the control pair is *ejected* from the target using the inverse of the injection weave, thereby returning the control qubit to its original state. As before, if the control qubit is in the state $|0_L\rangle$ the result is the identity. However, if the control qubit is in the state $|1_L\rangle$, a NOT gate is performed on the target qubit. This construction therefore produces a controlled-NOT gate, up to single-qubit rotations [27]. Because a weave producing any single-qubit rotation can be used instead of the NOT weave shown in Fig. 4(b) this construction can be used to produce an arbitrary controlled-rotation gate.

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