Stability of an Impulsively Accelerated Density Interface in Magnetohydrodynamics

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In the framework of ideal incompressible magnetohydrodynamics, we examine the stability of an impulsively accelerated, sinusoidally perturbed density interface in the presence of a magnetic field that is parallel to the acceleration. This is accomplished by analytically solving the linearized initial value problem, which is a model for the Richtmyer-Meshkov instability. We find that the initial growth rate of the interface is unaffected by the presence of a magnetic field, but for a finite magnetic field the interface amplitude asymptotes to a constant value. Thus the instability of the interface is suppressed. The interface behavior from the analytical solution is compared to the results of both linearized and nonlinear compressible numerical simulations.

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The Richtmyer-Meshkov instability (RMI) is important in a wide variety of applications [1] including inertial confinement fusion [2] and astrophysical phenomena [3]. In these applications, the fluids involved may be ionized and hence be affected by magnetic fields. Samtaney [4] has demonstrated, via numerical simulations, that the growth of the RMI is suppressed in the presence of a magnetic field. The particular flow studied was that of a shock interacting with an oblique planar contact discontinuity (CD) separating conducting fluids of different densities within the framework of planar ideal magnetohydrodynamics (MHD). It was shown that the suppression of the instability is caused by changes in the shock refraction process at the CD with the application of a magnetic field [5]. These changes prevent the deposition of circulation on the CD.

A more widely studied flow results from a shock wave accelerating a density interface with a single-mode sinusoidal perturbation in amplitude. Our goal is to understand the effect of a magnetic field on this flow when conducting fluids are involved. The magnetic field is again aligned with the motion of the shock. The initial condition for this flow is illustrated in Fig. 1(a). It is characterized by the incident shock sonic Mach number, M, the density ratio across the CD, ρ_2/ρ_1, the ratio of the CD's initial amplitude to its wavelength, η_0/λ, the ratio of specific heats, γ, and the nondimensional strength of the applied magnetic field, $\beta^{-1} = B^2/(2p_0)$. Here B is the magnitude of the applied magnetic field and p_0 is the initial pressure in the unshocked regions of the flow. As a model for this flow, we will examine the growth of a sinusoidally perturbed interface separating incompressible conducting fluids that is impulsively accelerated at $t = 0$. The setup for the model problem is illustrated in Fig. 1(b). This problem is characterized by ρ_1/ρ^*, ρ_2/ρ^*, η_0/λ, β, and the normalized magnitude of the impulse, $V\sqrt{\rho^*/p_0}$. We choose ρ^* to be ρ_1 from the corresponding shock driven flow.

In this investigation, it is convenient to consider solutions to the linearized equations of ideal, incompressible MHD in a noninertial reference frame that has acceleration $V\delta(t)$ in the z direction. Here, $\delta(t)$ is the Dirac delta function and $V \ll c$. The equations are linearized about a base flow that results from the impulsive acceleration of an unperturbed interface. This flow has no x dependence and zero vertical velocity (u). Our choice of reference frame results in the horizontal velocity (w) being zero for all time. The complete base flow is thus

$$\rho_0(z) = \rho_1 + H(z)(\rho_2 - \rho_1), \qquad u_0 = 0, \qquad w_0 = 0,$$
$$B_{x0} = 0, \qquad B_{z0} = B,$$
$$p_0(z, t) = -\rho_1 V\delta(t)z - H(z)(\rho_2 - \rho_1)V\delta(t)z,$$

where $H(z)$ is the Heaviside function, ρ is the density, p is the pressure, $\mathbf{u}$ is the velocity, and $\mathbf{B}$ is the magnetic field. When the interface is perturbed, the density becomes $\rho_0(z-h)$, where $h(x,t)$ is the position of the interface and $h \ll \lambda$. The linearized equations are obtained by assuming that all flow quantities, except density, are of the form $q(x,z,t) = q_0(z) + q'(x,z,t)$, where q' are small perturbations to the base flow. These expressions are then substituted into the governing equations. Neglecting terms involving products of perturbations, the resulting linearized equations are

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \tag{1}$$

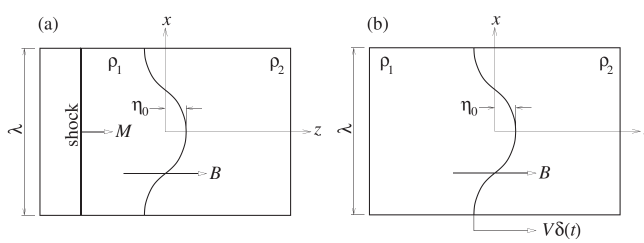

FIG. 1. (a) Initial condition geometry for compressible RMI. (b) Geometry for incompressible model problem.

$$\rho\frac{\partial u'}{\partial t}+\frac{\partial p'}{\partial x}=B\left(\frac{\partial B'_x}{\partial z}-\frac{\partial B'_z}{\partial x}\right), \quad (2)$$

$$\rho\frac{\partial w'}{\partial t}+\frac{\partial p'}{\partial z}=(\rho_2-\rho_1)[H(z)-H(z-h)]V\delta(t), \quad (3)$$

$$\frac{\partial B'_x}{\partial x}+\frac{\partial B'_z}{\partial z}=0, \quad (4)$$

$$\frac{\partial \mathbf{B}'}{\partial t}=B\frac{\partial \mathbf{u}'}{\partial z}. \quad (5)$$

Note that the forcing resulting from the impulse is nonzero only in a small region between $z=0$ and the interface. We assume that all perturbations have the form $q'(x,z,t)=\hat{q}(z,t)e^{ikx}$. We take our initial conditions to be at $t=0^-$, just prior to the impulsive acceleration, when the velocity and magnetic field perturbations are zero. Taking the temporal Laplace transforms of (1)–(5) outside of the forced region in each fluid gives

$$ikU_i+DW_i=0, \quad (6)$$

$$s\rho_iU_i+ikP_i=B[DH_{x_i}-ikH_{z_i}], \quad (7)$$

$$s\rho_iW+DP_i=0, \quad (8)$$

$$ikH_{x_i}+DH_{z_i}=0, \quad (9)$$

$$sH_{x_i}=BDU_i, \quad (10)$$

$$sH_{z_i}=BDW_i, \quad (11)$$

where U, W, H_x, H_z, and P are the temporal Laplace transforms of $\hat{u}$, $\hat{w}$, $\hat{B}_x$, $\hat{B}_z$, and $\hat{p}$, respectively, $i=1$ or 2, and $D\equiv\partial/\partial z$. Combining (6)–(11), we obtain

$$\left(D^2-\frac{\rho_i s^2}{B^2}\right)(D^2-k^2)W_i=0, \quad (12)$$

which has the general solution

$$W_i=A_ie^{kz}+B_ie^{-kz}+C_ie^{sz/C_{Ai}}+D_ie^{-sz/C_{Ai}}, \quad (13)$$

where $C_{Ai}=B/\sqrt{\rho_i}$ is the Alfvén wave speed in fluid i and the coefficients are functions of s. The inverse Laplace transforms (ILTs) of the first two terms have the form $f(t)e^{\pm kz}$ while the ILTs of the last two terms have the form $H(t\pm z/C_{Ai})f(t\pm z/C_{Ai})$. This causes the solution to be nonseparable in z and t.

Solutions to (1)–(5) are subject to a number of boundary conditions. The perturbations must be bounded as $|z|\to\infty$; thus $A_2(s)=0$ and $B_1(s)=0$. Also, we require that there be no incoming waves from $z=\pm\infty$; thus $C_2(s)=0$ and $D_1(s)=0$. Note that we have assumed $B>0$ and $k>0$. The resulting expressions for W_1 and W_2 are

$$W_1(z,s)=A_1(s)e^{kz}+C_1(s)e^{sz/C_{A1}}, \quad (14)$$

$$W_2(z,s)=B_2(s)e^{-kz}+D_2(s)e^{-sz/C_{A2}}. \quad (15)$$

At the contact $[z=h(x,t)=\eta(t)e^{ikx}]$, w', B'_x, and B'_z must be continuous (see pages 458–459 in [6]). Taking the Laplace transforms of these variables and using (6)–(11) to express each in terms of W, these boundary conditions become, to leading order in h,

$$[W]_{z=0}=0\to A_1+C_1=B_2+D_2, \quad (16)$$

$$[DW]_{z=0}=0\to kA_1+\frac{sC_1}{C_{A1}}=-kB_2-\frac{sD_2}{C_{A2}}, \quad (17)$$

$$[D^2W]_{z=0}=0\to k^2A_1+\frac{s^2C_1}{C_{A1}^2}=k^2B_2+\frac{s^2D_2}{C_{A2}^2}, \quad (18)$$

where $[q]_{z=0}\equiv q_2|_{z=0}-q_1|_{z=0}$ and (14) and (15) were used to obtain the expressions on the right.

The final boundary condition is derived by integrating (3) with regard to z from 0 to $h(x,t)$, across the inhomogeneous region. Neglecting higher order terms in h and using the fact that p' is continuous across the contact, this gives

$$p'_2(x,0,t)-p'_1(x,0,t)=(\rho_2-\rho_1)V\delta(t)\eta(t)e^{ikx}.$$

Taking the Laplace transform of this equation and using (6), (7), (10), (11), (14), and (15), the final boundary condition can be expressed as

$$\rho_1\left(\frac{sA_1}{k}+\frac{BC_1}{\sqrt{\rho_1}}\right)+\rho_2\left(\frac{sB_2}{k}+\frac{BD_2}{\sqrt{\rho_2}}\right)=(\rho_2-\rho_1)V\eta_0. \quad (19)$$

Equations (16)–(19) are solved for the four unknown coefficients: $A_1(s)$, $B_2(s)$, $C_1(s)$, and $D_2(s)$. The inverse Laplace transforms of these can be expressed as

$$a_1(t)=K_A\left[\frac{2\alpha_1^2e^{\alpha_1 t}}{(\alpha_1-\sigma)^2+\tau^2}+\Re\left(\frac{(\sigma+i\tau)(\alpha_1+\sigma+i\tau)e^{(\sigma+i\tau)t}}{i\tau(\sigma+i\tau-\alpha_1)}\right)\right],$$

$$b_2(t)=K_A\left[\frac{2\alpha_2^2e^{\alpha_2 t}}{(\alpha_2-\sigma)^2+\tau^2}+\Re\left(\frac{(\sigma+i\tau)(\alpha_2+\sigma+i\tau)e^{(\sigma+i\tau)t}}{i\tau(\sigma+i\tau-\alpha_2)}\right)\right],$$

$$c_1(t)=K_C\left[\frac{(\alpha_1+\alpha_2)e^{\alpha_1 t}}{(\alpha_1-\sigma)^2+\tau^2}+\Re\left(\frac{(\alpha_2+\sigma+i\tau)e^{(\sigma+i\tau)t}}{i\tau(\sigma+i\tau-\alpha_1)}\right)\right],$$

$$d_2(t)=K_D\left[\frac{(\alpha_1+\alpha_2)e^{\alpha_2 t}}{(\alpha_2-\sigma)^2+\tau^2}+\Re\left(\frac{(\alpha_1+\sigma+i\tau)e^{(\sigma+i\tau)t}}{i\tau(\sigma+i\tau-\alpha_2)}\right)\right],$$

where

$$\alpha_1=\frac{Bk}{\sqrt{\rho_1}},\qquad \alpha_2=\frac{Bk}{\sqrt{\rho_2}},$$

$$\sigma=-\frac{Bk(\sqrt{\rho_1}+\sqrt{\rho_2})}{\rho_1+\rho_2},$$

$$\tau=\frac{[B^2k^2(\rho_1+\rho_2-2\sqrt{\rho_1\rho_2})]^{1/2}}{\rho_1+\rho_2},$$

and

$$K_A=kV\eta_0\mathcal{A},\quad K_C=-\frac{2Bk^2V\eta_0\mathcal{A}\rho_2}{\sqrt{\rho_1}\rho_2+\sqrt{\rho_2}\rho_1},\quad K_D=\frac{\rho_1}{\rho_2}K_C.$$

The Atwood number $\mathcal{A} \equiv (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$. The above expressions are not valid if $\tau = 0$, but this requires that either $B = 0$, $k = 0$, or $\rho_1 = \rho_2$, which correspond to cases that are not of interest here. In the general case, from (14) and (15), the complete solutions for w in each fluid are

$$w_1 = [H(t)a_1(t)e^{kz} + H(t + z/C_{A1})c_1(t + z/C_{A1})]e^{ikx},$$
$$w_2 = [H(t)b_2(t)e^{-kz} + H(t - z/C_{A2})d_2(t - z/C_{A2})]e^{ikx}.$$

The exponents $\alpha_1 t$ and $\alpha_2 t$ are positive, admitting the possibility that the maximum velocity grows exponentially in time. This does not occur for the following reasons. First consider fluid 2. For $0 < z < C_{A2}t$, it can be shown that the terms in w_2' involving the exponent $\alpha_2 t$ cancel. For $z > C_{A2}t$, the term involving the exponent $\alpha_2 t$ has the form

$$Ke^{\alpha_2 t - kz}e^{ikx} = Ke^{-k(z - C_{A2}t)}e^{ikx}. \tag{20}$$

This term decays exponentially in the moving coordinate $z - C_{A2}t$, which is positive for $z > C_{A2}t$. Thus the maximum of w_2' does not grow exponentially in time. Similar arguments hold in fluid 1.

Two fronts that propagate at the local Alfvén speed arise naturally in the solution. At their locations, the solution satisfies the appropriate linearized MHD Rankine-Hugoniot relations.

The solution shows that the initial ($t = 0^+$) growth rate of the interface is unaffected by the presence of a magnetic field, specifically

$$\left.\frac{\partial \eta}{\partial t}\right|_{t=0^+} = \eta_0 k V \mathcal{A}, \tag{21}$$

as in the hydrodynamic case [7].

Profiles of $\hat{w}(z, t)$ at various times are shown in Fig. 2 for one set of parameters. The value of $\hat{w}(z, t)$ at $z = 0$ is the growth rate of the interface. From Fig. 2, it can be seen that as t increases and the Alfvén fronts propagate away from the interface, carrying away the majority of the vorticity produced by the impulsive acceleration, the growth rate of the interface decays to zero. Thus the instability of the interface is suppressed and its amplitude asymptotes to a constant value. For $t \to \infty$, the interface amplitude tends to

$$\eta_\infty = \eta_0[1 + V(C_{A2}^{-1} - C_{A1}^{-1})]. \tag{22}$$

This shows that the change in interface amplitude is inversely proportional to B. Thus for $B \to 0$, $\eta_\infty \to \infty$, which is in agreement with the result from hydrodynamic linear stability analysis [7]. Interestingly, η_∞ is independent of wave number.

We have compared the preceding solution with the results of three different numerical simulations. The first simulation was carried out with a nonlinear compressible MHD solver that uses the 8-wave upwinding formulation [8] with an unsplit upwinding method [9]. The solenoidal property of the magnetic field is enforced at each time step using a projection method. In this simulation, the interface is accelerated by a shock as depicted in Fig. 1(a). The second and third simulations were carried out using a method for obtaining numerical solutions to the linearized MHD equations wherein the base state is temporally evolving. In this method, we begin by writing the equations of compressible MHD in conservative form in two dimensions as follows.

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial H(U)}{\partial z} = 0, \tag{23}$$

where $U \equiv U(x, z, t) = \{\rho, \rho u, \rho v, \rho w, B_x, B_y, B_z, e\}^T$, and the vectors $F(U)$ and $H(U)$ are the fluxes of mass, momentum, magnetic field, and total energy in the x and z directions, respectively. In the above equations, e is the total energy per unit volume. Writing the solution as $U(x, z, t) = U^0(z, t) + \epsilon\hat{U}(z, t)\exp(ikx)$, where $U^0(z, t)$ is a one-dimensional temporally evolving base state, and $\hat{U}(z, t)\exp(ikx)$ is the perturbation, we get

$$\frac{\partial U^0}{\partial t} + \frac{\partial H(U^0)}{\partial z} = 0, \tag{24}$$

$$\frac{\partial \hat{U}}{\partial t} + \frac{\partial A(U^0)\hat{U}}{\partial z} = ikB(U^0)\hat{U}, \tag{25}$$

where $B(U^0)$ is the Jacobian of $F(U^0)$ with respect to U^0. Equation (24) governs the evolution of the base state, while the perturbations are governed by (25), where $A(U^0)$ is the Jacobian of $H(U^0)$ with respect to U^0. We employ a third order TVD Runge-Kutta time integration method, and adopt a finite volume upwind approach wherein the fluxes are calculated using Roe's method. Details of the numerical method are presented in a separate publication [10].

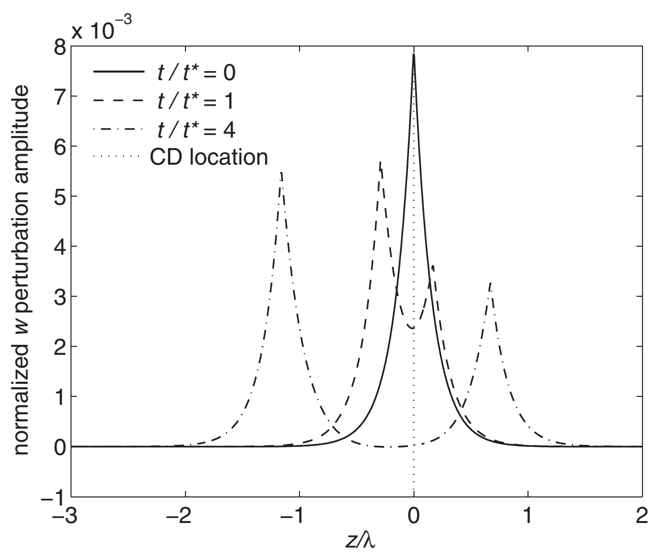

FIG. 2. Profiles of $\hat{w}(z, t)\sqrt{\rho^*/p_0}$ at $t/t^* = 0$, $t/t^* = 1$, and $t/t^* = 4$, for $\rho_1/\rho^* = 1.48372$, $\rho_2/\rho^* = 4.43159$, $V\sqrt{\rho^*/p_0} = 0.319\,125$, $\eta_0/\lambda = 0.00\,799\,276$, and $\beta = 16$. Here $t^* \equiv \lambda\sqrt{\rho^*/p_0}$. The maxima of $\hat{w}(z, t)$ coincide with the Alfvén fronts.

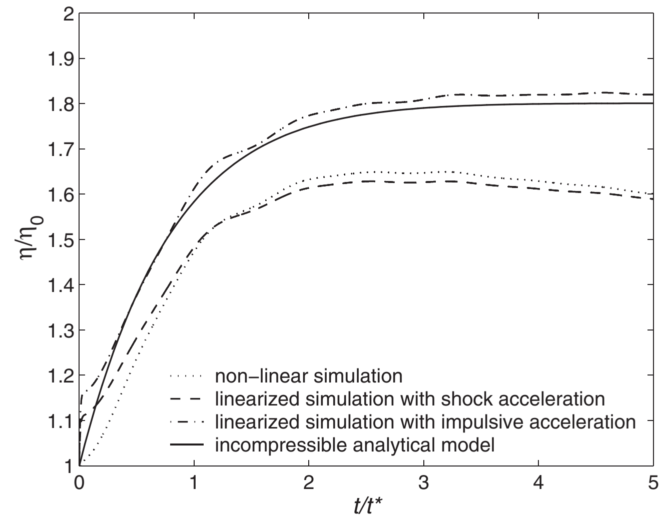

FIG. 3. Interface amplitude histories from the current model and a compressible linearized simulation with an impulsive acceleration, both with $\rho_1/\rho^* = 1.48372$, $\rho_2/\rho^* = 4.43159$, $V\sqrt{\rho^*/p_0} = 0.319\,125$, $\eta_0/\lambda = 0.00\,799\,276$, and $\beta = 16$, and both linearized and nonlinear compressible simulations with a shock accelerated interface with $M = 1.25$, $\beta = 16$, $\rho_2/\rho_1 = 3$, $\eta_0/\lambda = 0.01$, and $\gamma = 5/3$.

The base flow for the second simulation is the solution to the Riemann problem that arises from the interaction of a shock with an unperturbed density interface. Hence, the density interface in the second simulation is also shock accelerated. In the third simulation the interface is impulsively accelerated. The magnitude of the impulse is set to the velocity of the interface in the Riemann problem solution that forms the base flow for the second simulation. The initial densities and pressures of the two fluids in the third simulation are set to the values immediately on either side of the interface in the Riemann problem solution. The initial perturbation amplitude of the interface is taken from the first simulation immediately after the interface has been compressed by the passage of the shock wave. The same initial conditions are used in the model. Figure 3 shows the interface amplitude histories from the model and the three simulations that approximate a shock accelerated interface with $M = 1.25$, $\beta = 16$, $\rho_2/\rho_1 = 3$, $\eta_0/\lambda = 0.01$, and $\gamma = 5/3$. There is close agreement between the behavior of the interface predicted by the model and the third simulation, with the final interface amplitudes being within approximately 1% of each other. Small amplitude oscillations can be seen in the simulation result. These are caused by the additional waves present in the simulation because it is compressible. Comparing the results of the second and third simulations shows the effect of the interface being shock accelerated rather than impulsively accelerated. The qualitative behavior of the interface is similar in both cases, but the shock acceleration appears to result in significantly less growth of the interface amplitude. This is also evident in the results from the nonlinear simulation.

In conclusion, we have examined the behavior of an impulsively accelerated perturbed interface separating incompressible conducting fluids of different densities, in the presence of a magnetic field that is parallel to the acceleration. This was done by analytically solving the appropriate linearized initial value problem. We find that the initial growth rate of the interface is unaffected by the presence of a magnetic field. The growth rate then decays resulting in the interface amplitude asymptoting to a constant value. The difference between the initial and final interface amplitudes is inversely proportional to the magnetic field magnitude. Thus the instability of the interface is suppressed by the presence of the magnetic field. For the set of parameters considered here, the interface behavior given by the analytical solution well approximates that seen in the results of a compressible linearized simulation in which the interface is impulsively accelerated. Acceleration of the interface by a shock results in significantly less growth of the interface amplitude.

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