

New Type of Magnetorotational Instability in Cylindrical Taylor-Couette Flow

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We study the stability of cylindrical Taylor-Couette flow in the presence of combined axial and azimuthal magnetic fields, and show that adding an azimuthal field profoundly alters the previous results for purely axial fields. For small magnetic Prandtl numbers Pm , the critical Reynolds number Re_c for the onset of the magnetorotational instability becomes independent of Pm , whereas for purely axial fields it scales as Pm^{-1} . For typical liquid metals, Re_c is then reduced by several orders of magnitude, enough that this new design should succeed in realizing this instability in the laboratory.

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The magnetorotational instability (MRI) is one of the most important processes in astrophysics, with applications ranging from accretion disks to entire galaxies [1,2]. There is therefore great interest in trying to study this instability in the laboratory [3–10]. These experiments are complicated by the extremely small magnetic Prandtl numbers of liquid metals, which necessitate very large rotation rates. Here we report on a new type of magnetorotational instability, that operates even at infinitesimal Prandtl number, and correspondingly at far lower rotation rates. This new instability should be much easier to obtain experimentally, and may also have considerable astrophysical implications.

The magnetorotational instability is a mechanism for transporting angular momentum. Consider, for example, an accretion disk around a newly forming star. In order for material to fall inward, it must transfer its angular momentum to material further out. The difficulty is how to accomplish this. In particular, a Keplerian angular velocity profile, for which $\Omega \sim r^{-3/2}$, is known to be hydrodynamically stable (at least linearly), by the familiar Rayleigh criterion. Purely viscous coupling in a laminar flow is many orders of magnitude too small though; if this were the angular momentum transport mechanism, stars would take so long to form that none would yet exist today, some 14×10^9 years after the big bang.

This conundrum was solved by Balbus and Hawley [1], who showed that such a Keplerian profile may be hydrodynamically stable, but is nevertheless magnetohydrodynamically unstable. The addition of a magnetic field opens up a new way of coupling fluid parcels, namely, via the magnetic tension in the field lines, and thereby allows for a new instability, the magnetorotational instability, that has no analog in the purely hydrodynamic problem. Coupling the fluid in this way is then so much more efficient at transporting angular momentum that this mechanism can indeed yield accretion rates more in line with those observed. Something as basic as the time it takes for a star to form is thus magnetically controlled.

Subsequent to this work, it was soon realized that this instability had actually been discovered decades earlier, by Velikhov [11], who had, though, not considered it in an astrophysical context. Instead, he had viewed it simply as the magnetohydrodynamic extension of the so-called Taylor-Couette problem, consisting of the flow between differentially rotating cylinders. See, for example, [12] for a review of this problem, one of the oldest in fluid dynamics.

Once the connection is made to the classical Taylor-Couette problem, one immediately recognizes that this should be a way to study this instability in the lab: just take the fluid to be a liquid metal, and apply a magnetic field along the cylinders. This simple design is the basis of most of the MRI experiments proposed to date [3–8]. (Sisan *et al.* [9] claim to have obtained the magnetorotational instability already, in spherical rather than cylindrical geometry. However, there are enough other instabilities that can arise in this configuration, e.g., [13], that their results are also open to other interpretations. Indeed, the basic state from which their instability arises is already fully turbulent, indicating that whatever their instability may be, it is certainly not the first instability to set in.)

As simple as it sounds, there is unfortunately also one considerable difficulty associated with these experiments, namely, that the rotation rates of the inner and outer cylinders must be enormous. The problem is that the relevant parameter is not so much the hydrodynamic Reynolds number $Re = \Omega_i r_i^2 / \nu$, but rather the magnetic Reynolds number $Rm = \Omega_i r_i^2 / \eta$, where ν is the viscosity and η the magnetic diffusivity. In its traditional form, the magnetorotational instability sets in when $Rm = O(10)$. Re is then given by Rm/Pm , where $Pm = \nu/\eta$ is the magnetic Prandtl number, a material property of the fluid. Typical values are $\sim 10^{-5}$ for liquid sodium, $\sim 10^{-6}$ for gallium, and $\sim 10^{-7}$ for mercury. Re must therefore be at least 10^6 , and possibly larger still, depending on the particular liquid metal one intends to use.

Such large values can be reached in the lab; taking the inner cylinder radius $r_i = 10$ cm, say, one finds that rotation rates of around 10 Hz are required—large, but achievable. The difficulty lies elsewhere, namely, in the ends that would necessarily be present in any real experiment. Because of the Taylor-Proudman theorem, stating that in rapidly rotating systems the flow will tend to align itself along the axis of rotation, end effects become increasingly important, until at $\text{Re} = O(10^6)$ the flow is controlled almost entirely by the end plates [14]. That is, while it is possible to achieve such large rotation rates, the flow will look nothing like the idealized infinite-cylinder flow on which all of the theoretical analysis is based.

A new approach is therefore needed if this instability is to be studied in the laboratory. We propose the following: instead of imposing only a uniform axial field $B_z = B_0$, additionally impose an azimuthal field $B_\phi = \beta B_0(r_i/r)$, where β is a nondimensional measure of the relative magnitudes of B_ϕ and B_z , and the dimensional quantity B_0 will be incorporated into the Hartmann number below. Such a B_ϕ field can be generated just as easily as a B_z field, by running a current-carrying wire down the axis of the inner cylinder (suitably insulated from the fluid, of course). Note also that with neither B_z nor B_ϕ maintained by currents within the fluid itself, the possibility of magnetic instabilities is excluded *a priori*. The only source of energy to drive an instability is the imposed differential rotation; the magnetic field merely acts as a catalyst.

Given this basic state consisting of these externally imposed magnetic fields, as well as the differential rotation profile $\Omega(r)$ imposed by the rotation rates Ω_i and Ω_o of the two cylinders, we linearize the governing equations about it, and look for axisymmetric disturbances. These are known to be preferred over nonaxisymmetric ones for the classical $\beta = 0$ MRI, and are therefore the appropriate starting point here as well. The perturbation flow and field may then be expressed as

$$\mathbf{u} = v\hat{\mathbf{e}}_\phi + \nabla \times (\psi\hat{\mathbf{e}}_\phi), \quad \mathbf{b} = b\hat{\mathbf{e}}_\phi + \nabla \times (a\hat{\mathbf{e}}_\phi).$$

Taking the z and t dependence to be $\exp(ikz + \gamma t)$, the perturbation equations become

$$\text{Re}\gamma v = D^2 v + \text{Re}ikr^{-1}(r^2\Omega)'\psi + \text{Ha}^2 ikb,$$

$$\text{Re}\gamma D^2 \psi = D^4 \psi - \text{Re}2ik\Omega v + \text{Ha}^2 ik(D^2 a + 2\beta r^{-2}b),$$

$$\text{Pm Re}\gamma b = D^2 b - \text{Pm Re}ik\Omega'ra + ikv - 2ik\beta r^{-2}\psi,$$

$$\text{Pm Re}\gamma a = D^2 a + ik\psi,$$

where $D^2 = \nabla^2 - 1/r^2$, and the primes denote d/dr .

Length has been scaled by r_i , time by Ω_i^{-1} , Ω by Ω_i , \mathbf{u} by η/r_i , and \mathbf{B}_0 and \mathbf{b} by B_0 . The various nondimensional parameters are then: (a) the magnetic Prandtl number $\text{Pm} = \nu/\eta$ already mentioned above, (b) the ratio $\hat{\mu} = \Omega_o/\Omega_i$ [this enters into the details of $\Omega(r) = c_1 + c_2/r^2$] and the Reynolds number $\text{Re} = \Omega_i r_i^2/\nu$, measuring the relative and absolute rotation rates of the two cylinders,

(c) the parameter β and the Hartmann number $\text{Ha} = B_0 r_i / \sqrt{\rho \mu \eta \nu}$ [ρ is the fluid's density, μ the magnetic permeability], measuring the relative and absolute magnitudes of the imposed magnetic fields, and (d) the radius ratio r_i/r_o , which we fixed at $1/2$.

The radial structure of v , ψ , b , and a was expanded in terms of Chebyshev polynomials, typically up to $N = 40$ – 80 . These equations and associated boundary conditions (no slip for \mathbf{u} , insulating for \mathbf{b}) then reduce to a large ($4N \times 4N$) matrix eigenvalue problem, with the eigenvalue being the growth or decay rate γ of the given mode. Note also that this numerical implementation is very different from that of [4], in which the individual components of \mathbf{u} and \mathbf{b} were used, and discretized in r by finite differencing. Both codes yielded identical results though in every instance where we benchmarked one against the other.

The following sequence of calculations was then carried out. First, we fixed Pm , β , and $\hat{\mu}$, and scanned through a range of values of Re , Ha , and k , in each case determining whether the given modes grow or decay. We thus found the smallest value of Re , and the corresponding Ha and k , that yields a marginally stable solution, one having $\text{Re}(\gamma) = 0$ (which typically involved solving the basic eigenvalue problem for several thousand combinations of Re , Ha , and k). By repeating this entire procedure for different values of Pm , β , and $\hat{\mu}$, we obtained the results in Fig. 1.

The dotted curve to the left of $\hat{\mu} = 0.25$ is a purely hydrodynamic instability, namely, the onset of Taylor vortices, e.g., [12]. As we approach $\hat{\mu} = 0.25$ though, we note that $\text{Re}_c \rightarrow \infty$. This critical value $\hat{\mu} = 0.25$ [$(r_i/r_o)^2$, in general] is precisely the so-called Rayleigh line, beyond which the flow is hydrodynamically stable, because the angular momentum increases outward.

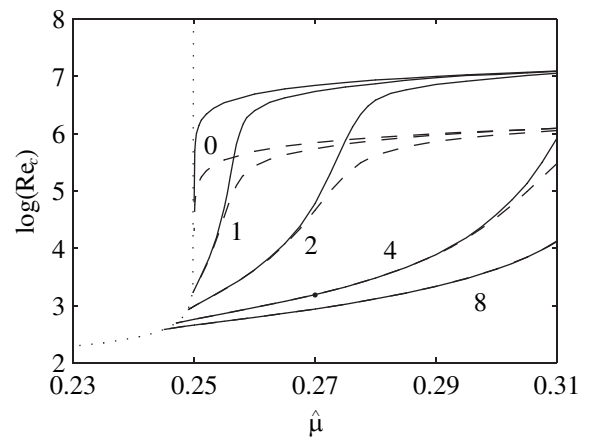


FIG. 1. The critical Reynolds number Re_c as a function of $\hat{\mu}$. The dotted curve to the left of $\hat{\mu} = 0.25$ is a purely nonmagnetic instability ($\text{Ha} = 0$, so b and a drop out entirely). The curves to the right of $\hat{\mu} = 0.25$ are the magnetorotational instability. Solid curves are $\text{Pm} = 10^{-6}$, dashed curves $\text{Pm} = 10^{-5}$. The numbers beside pairs of curves indicate β . Finally, the dot on the $\beta = 4$ curves corresponds to the solution shown in Fig. 2.

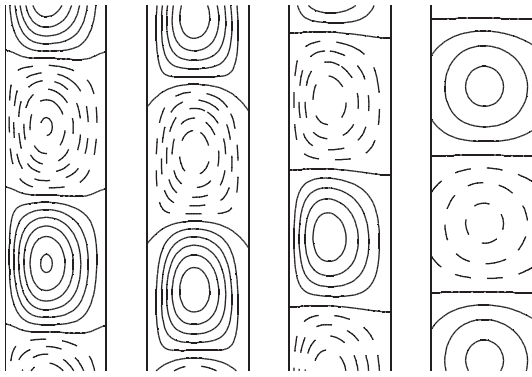


FIG. 2. The spatial structure of the eigenmode for $\text{Pm} = 0$, $\beta = 4$, and $\hat{\mu} = 0.27$, for which $\text{Re}_c = 1521$ (see Fig. 1), $\text{Ha} = 16.3$ and $k = 2.33$. From left to right, contours of v , ψ , b , and a , with the contour interval for b and a $1/3$ that of v and ψ . Note the absence of any special phase relationship between the various quantities, due to the $\pm z$ symmetry breaking.

We are more interested therefore in the behavior to the right of $\hat{\mu} = 0.25$, where we anticipate that the inclusion of magnetic effects will yield the MRI. We begin with the two curves $\beta = 0$, the MRI as it has been considered to date [10]. We see how Re_c rises very steeply from the previous nonmagnetic results to the left of $\hat{\mu} = 0.25$, and then scales as Pm^{-1} , exactly as described above, and in perfect agreement with [4]. And again, because Pm is so small, these values end up being too large for the experiment to succeed [14].

However, turning next to the results for nonzero β , we see that Re_c is dramatically reduced, over a range of $\hat{\mu}$ extending increasingly far beyond the Rayleigh line. For example, if we focus on how far $\hat{\mu}$ can be increased before $\text{Re}_c = 10^4$, say, we obtain $\hat{\mu} = 0.253, 0.264, 0.292$, and 0.308 for $\beta = 1, 2, 4$, and 8 , respectively. Furthermore, within these $\hat{\mu}$ ranges Re_c is independent of Pm , very different from the previous Pm^{-1} scaling. Indeed, for nonzero β , and up to these values for $\hat{\mu}$, one can set $\text{Pm} = 0$ and still obtain exactly the same solutions. This $\text{Pm} = 0$ limit was considered before by Chandrasekhar [15], but for axial fields only, in which case there are no instabilities to the right of the Rayleigh line.

The explanation for this very different behavior for nonzero β lies in the coupling between the azimuthal field b and the meridional circulation ψ . If $\beta = 0$ these quantities are not directly coupled at all, only indirectly through v and a . For nonzero β each acts directly on the other. Of these two new terms in the equations, the more important one turns out to be the effect of ψ on b . Physically, this corresponds to the meridional circulation ψ advecting the original B_ϕ and thereby generating a contribution to b . It is this new way of maintaining b that allows this instability to proceed even in the $\text{Pm} = 0$ limit, where the classical $\beta = 0$ MRI fails.

Figure 2 shows an example of these new solutions. We note that however dramatically Re_c may have been re-

duced, the spatial structure is much the same as for $\beta = 0$, consisting of Taylor vortices elongated slightly in the z direction. There is actually one subtle difference, namely, that the up or down symmetry in z has been broken. This is due to the handedness of the imposed field, which distinguishes between $\pm z$ in a way that a purely axial field does not. As a result of this symmetry breaking, these new modes are also no longer stationary, but instead have $\text{Im}(\gamma) \neq 0$, corresponding to a drift in z , at the rate $\text{Im}(\gamma)/k$. This particular solution has $\text{Im}(\gamma) = 0.153$ and $k = 2.33$, for a drift rate of $0.066(r_i\Omega_i)$.

Turning next to the Hartmann number $\text{Ha} = 16.3$, this translates into a ~ 12 G field, taking the fluid to be liquid sodium, and $r_i = 10$ cm. An axial field of that magnitude is certainly easily achievable in the lab. For the azimuthal field we then want 500 G cm/ r , corresponding to a current of 2500 A in this wire running down the central axis, which is again achievable, if perhaps not quite so easily.

To summarize then, we have shown that the magnetorotational instability is radically altered if one imposes both axial and azimuthal magnetic fields, becoming independent of the magnetic Prandtl number in the $\text{Pm} \rightarrow 0$ limit—a result that is all the more remarkable as neither purely axial nor purely azimuthal fields yield anything like it. With this new scaling, the critical Reynolds numbers are reduced by several orders of magnitude, enough that these instabilities could perhaps be attainable in the lab without being disrupted by end effects (although of course some end effects will always be present, particularly with this slow drift in z). Further computational work includes the nonlinear equilibration of these modes, as well as the possibility of nonaxisymmetric instabilities. These and other issues are currently being explored.

Finally, returning briefly to the original astrophysical motivation, we note that virtually all astrophysical bodies do in fact have both axial and azimuthal magnetic fields. These magnetic fields are typically not externally imposed though, but rather generated by electric currents flowing within the system itself. Self-consistently solving for both the large-scale fields as well as the small-scale instabilities is then far more complicated than our analysis here, but the basic ingredients are certainly there for this new type of magnetorotational instability to play a role.

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