

Virial Theorem and Universality in a Unitary Fermi Gas

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(Received 25 March 2005; published 13 September 2005)

Unitary Fermi gases, where the scattering length is large compared to the interparticle spacing, can have universal properties, which are independent of the details of the interparticle interactions when the range of the scattering potential is negligible. We prepare an optically trapped, unitary Fermi gas of ${}^6\text{Li}$, tuned just above the center of a broad Feshbach resonance. In agreement with the universal hypothesis, we observe that this strongly interacting many-body system obeys the virial theorem for an ideal gas over a wide range of temperatures. Based on this result, we suggest a simple volume thermometry method for unitary gases. We also show that the observed breathing mode frequency, which is close to the unitary hydrodynamic value over a wide range of temperature, is consistent with a universal hydrodynamic gas with nearly isentropic dynamics.

DOI: [10.1103/PhysRevLett.95.120402](https://doi.org/10.1103/PhysRevLett.95.120402)

PACS numbers: 03.75.Ss, 32.80.Pj

Universal behavior is believed to approximately describe a variety of strongly interacting Fermi systems, such as neutron stars [1–4] and resonantly interacting atomic Fermi gases [1]. Universal Fermi systems satisfy the unitary condition, where the zero-energy scattering length greatly exceeds the interparticle spacing, while the range of the scattering potential is negligible. Unitary conditions are produced in an optically trapped Fermi gas [5], by using a magnetic field to tune near a broad Feshbach resonance, where strong interactions are observed [6]. According to the universal hypothesis, the interparticle spacing sets the only natural length scale, and the system can exhibit universal thermodynamics [1,3,4,6–9]. If unitary Fermi gases satisfy this hypothesis, they can be used to test predictions in fields well outside of atomic physics.

Universality at low temperature has been tested in measurements of two-component, strongly interacting Fermi gases. The ratio, β , of the interaction energy to the local Fermi energy has been measured [6,10–13] and is believed to be a zero-temperature, universal parameter [1]. The measurements are in reasonable agreement with recent predictions [14,15]. Universality also may explain the small line shifts observed in radio-frequency spectroscopy measurements [16]. Universality at low temperature is further supported by the spatial profile [6,12,13] and breathing mode frequency [17–19] of a trapped, highly degenerate, unitary Fermi gas. However, for a Fermi gas tuned to a broad Feshbach resonance, there have been no model-independent tests of the universal hypothesis over a wide temperature range, and these tests impact all theoretical predictions of the thermodynamics.

In this Letter, we show theoretically and experimentally that a harmonically trapped, unitary ${}^6\text{Li}$ Fermi gas obeys the virial theorem for an ideal gas over a wide range of temperature, as predicted by universal thermodynamics. Although the gas is a strongly interacting, many-body system, we find that the mean square width of the cloud varies linearly with the total energy, as predicted by the

virial theorem. We also show that universal hydrodynamics, under isentropic conditions, may describe the behavior of the observed breathing mode frequency [19], which is near the zero-temperature unitary hydrodynamic value over a wide range of temperatures, and has no mean field shift.

We test the universal hypothesis in a highly degenerate, unitary Fermi gas of ${}^6\text{Li}$, confined in a CO_2 laser trap. This far-detuned trap provides a nearly harmonic potential, which is the same for all atoms, both paired and unpaired. We employ a 50:50 mixture of the two lowest spin-up and spin-down hyperfine states, and tune a bias magnetic field to 840 G, just above the center of a broad s -wave Feshbach resonance. Forced evaporation in the optical trap is then used to cool the gas, as described previously [6,13,17–19]. After evaporation, we obtain a total of $N = 2.0(0.2) \times 10^5$ atoms, and the cloud has a nearly zero-temperature Thomas-Fermi profile, as expected for a unitary gas at zero temperature [13].

From the measured trap frequencies we obtain $\omega_{\perp} = \sqrt{\omega_x \omega_y} = 2\pi \times 1696(10)$ Hz, $\omega_x/\omega_y = 1.107(0.004)$, and $\lambda = \omega_z/\omega_{\perp} = 0.045$. The typical Fermi temperature (at the trap center for a noninteracting gas) is $T_F = (3N)^{1/3} \hbar(\omega_x \omega_y \omega_z)^{1/3}/k_B \simeq 2.4 \mu\text{K}$, small compared to the final trap depth of $U_0/k_B = 35 \mu\text{K}$. The coupling parameter of the strongly interacting gas at $B = 840$ G is $k_F a \simeq -30.0$, where $\hbar k_F = \sqrt{2m k_B T_F}$ is the Fermi momentum, and $a = a(B)$ is the zero-energy scattering length estimated from the measurements of Bartenstein *et al.* [20].

We now show that universality requires such a strongly interacting, unitary Fermi gas to obey the virial theorem for a harmonically trapped ideal gas at all temperatures. According to the universal hypothesis, the thermodynamic functions that describe the gas must be independent of the interaction parameters and can depend only on the total density n and temperature T [8].

Consider first the local energy ΔE (kinetic and interaction energy) contained in a small volume ΔV of gas

centered at position \mathbf{x} in a harmonic trap. Assume that the volume ΔV contains a fixed number of atoms ΔN , so that $n = \Delta N/\Delta V$, where $\int d^3\mathbf{x}n(\mathbf{x}) = N$ is the total number of trapped atoms.

For such a unitary gas, the local energy must be of the general form,

$$\Delta E = \Delta N \epsilon_F(n) f_E \left[\frac{T}{T_F(n)} \right]. \quad (1)$$

Here, the natural energy scale for atoms of mass m is taken to be $k_B T_F(n) = \epsilon_F(n)$ with $\epsilon_F(n) \equiv \hbar^2(3\pi^2 n)^{2/3}/(2m)$. With this definition, $\epsilon_F(n)$ is the local Fermi energy corresponding to the density n and $T_F(n)$ is the corresponding local Fermi temperature. Note that for a zero-temperature ideal Fermi gas, we have $f_E = 3/5$, for a zero-temperature unitary gas $f_E = 3(1 + \beta)/5$, while for a classical gas, $f_E = (3/2)T/T_F(n)$.

The corresponding local entropy ΔS takes the form

$$\Delta S = \Delta N k_B f_S \left[\frac{T}{T_F(n)} \right], \quad (2)$$

where $k_B f_S$ is the average entropy per particle, which can contain normal and superfluid contributions.

The local pressure of the gas is readily determined from the relation $P = -[\partial(\Delta E)/\partial(\Delta V)]_{\Delta N, \Delta S}$. From Eq. (2), we see that holding the local entropy constant requires $f_S = \text{const}$, which in turn means that we hold the local reduced temperature constant in taking the derivative of ΔE with respect to volume ΔV . Hence, we need only to find the volume derivative of the local Fermi energy, which yields the local pressure,

$$P = \frac{2}{3} \mathcal{E}(n, T), \quad (3)$$

where the local energy density (total kinetic plus interaction energy per unit volume) is $\mathcal{E}(n, T) = n \epsilon_F(n) \times f_E[T/T_F(n)]$. Equation (3) relates the pressure and local energy density for the unitary gas in the same way as for an ideal, noninteracting homogeneous gas, although the energy densities are quite different. Equation (3) for a unitary gas was obtained previously in Ref. [8].

In mechanical equilibrium, the balance of the forces arising from the pressure P and trapping potential U yields

$$\nabla P(\mathbf{x}) + n(\mathbf{x}) \nabla U(\mathbf{x}) = 0. \quad (4)$$

Taking an inner product of Eq. (4) with \mathbf{x} and using $\mathbf{x} \cdot \nabla U(\mathbf{x}) = 2U(\mathbf{x})$ for a harmonic trap, one readily obtains $N\langle U \rangle = (3/2) \int d^3\mathbf{x} P(\mathbf{x})$, where $\langle U \rangle$ is the average potential energy per particle. Using $\int d^3\mathbf{x} \mathcal{E}(\mathbf{x}) = E - N\langle U \rangle$ and Eq. (3) then yields

$$N\langle U \rangle = \frac{E}{2}. \quad (5)$$

Hence, universality requires a unitary Fermi gas to obey the virial theorem for an ideal gas. Since the mean square size is $\propto \langle U \rangle$, Eq. (5) is equivalent to

$$\frac{\langle x^2(E) \rangle}{\langle x^2(E_0) \rangle} = \frac{E}{E_0}, \quad (6)$$

which we use to verify the theorem. Here E_0 is the ground state energy of the cloud.

Energy is first added to gas, always starting from the lowest temperatures, by abruptly releasing the cloud and then recapturing it after a short expansion time t_{heat} [13]. During the expansion time, the total kinetic and interaction energy is conserved. When the trapping potential $U(\mathbf{x})$ is reinstated, the potential energy of the expanded gas is larger than that of the initially trapped gas, increasing the total energy to $E(t_{\text{heat}})$, which is a known function of t_{heat} [13]. After waiting for the cloud to reach equilibrium, the sample is released from the trap. The mean square width $\langle x^2 \rangle$ is estimated by fitting a one-dimensional, finite-temperature, Thomas-Fermi profile to the spatial distribution of the cloud [13], which is imaged after a *fixed* expansion time of 1 ms.

Figure 1 shows $\langle x^2 \rangle$ as a function E . The dashed line shows the fit, $\langle x^2 \rangle / \langle x^2(0) \rangle = 1.03(0.02)E/E_0$, which is in close agreement with the virial theorem prediction of Eq. (6) for a unitary gas. Equation (5) is therefore verified because $N\langle U(T=0) \rangle = E_0/2$, which follows generally from the equation of state for a unitary gas at zero temperature [6,10,13]. The Fermi radius σ'_x of the trapped unitary cloud is measured from a fit at nearly zero temperature [13], and determines $\langle x^2(T=0) \rangle = \sigma_x'^2/8$ as well as $E_0/N = 3m\omega_x^2 \langle x^2(T=0) \rangle$.

From these results, we see that despite the strong, many-body interactions, the total potential energy of the unitary

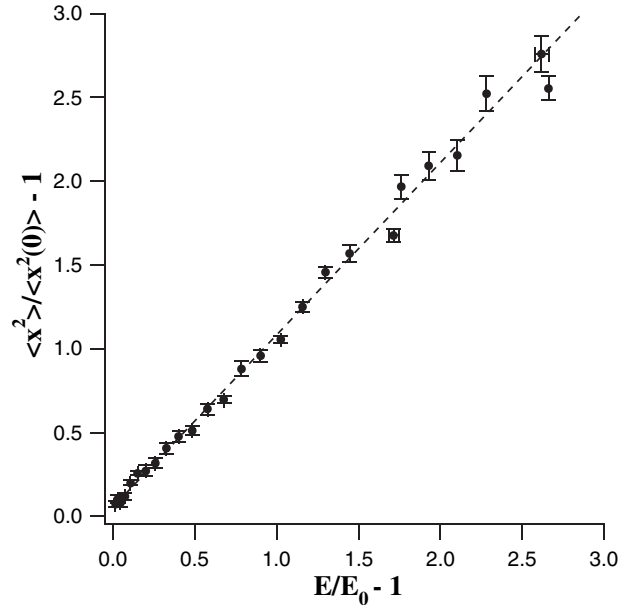


FIG. 1. Verifying the virial theorem in a unitary Fermi gas of ${}^6\text{Li}$: $\langle x^2 \rangle / \langle x^2(0) \rangle$ versus E/E_0 showing linear scaling. Here $\langle x^2 \rangle$ is the measured transverse mean square size. E is the total energy, calculated as in Ref. [13]. E_0 and $\langle x^2(0) \rangle$ denote ground state values.

gas is half of the total energy. Hence, the sum of the kinetic and interaction energies must be exactly half of the total energy at all temperatures.

We suggest that an empirical thermometry method can be based on the virial theorem result for a unitary Fermi gas. One can simply measure $\langle x^2 \rangle$ for a unitary gas near a broad Feshbach resonance. This determines the total energy according to Eq. (6), i.e., since $E/N = 3m\omega_x^2 \langle x^2 \rangle$. Universality then requires that E/E_0 is in one-to-one correspondence with the reduced temperature, T/T_F .

An approximate empirical reduced temperature can be determined by assuming that E/E_0 obeys ideal gas scaling with reduced temperature. At a later time, the empirical temperature can be calibrated, either theoretically or experimentally.

A theoretical calibration, relating the empirical and theoretical reduced temperatures, can be accomplished using an exact calculation of $E(T/T_F)/E_0$ for a unitary gas. While this calibration method is not useful for measurements of the energy versus temperature (where it is a tautology), it can be used to estimate temperature for precise quantitative comparisons between predictions and measurements of condensed pair fractions [21,22], the gap [23], collective mode damping rates [17,19], etc.

An experimental consistency check of the calibration can be done by measuring the total entropy of the unitary gas versus energy, $S(E)$. E is known from $\langle x^2 \rangle$. S can be determined, albeit in a model-dependent way at present [13,24], using an adiabatic sweep of the magnetic field [24] from the unitary regime to a weakly interacting regime either well below [22,23] or well above resonance [21], where S is calculated [13,24]. Then, the fundamental relation, $1/T = (\partial S/\partial E)$, yields the temperature of the unitary gas for a given E , which can be compared to the empirical temperature measured at the corresponding E .

We now consider the implications of universal hydrodynamics under isentropic expansion conditions, which may apply to the radial breathing mode of a unitary Fermi gas [19]. We find that the damping rate reveals a transition in behavior, while the frequency remains close to the zero-temperature unitary hydrodynamic value over a wide range of temperature [19].

Under locally isentropic conditions, the stream velocities of the normal and superfluid components must be equal (since the entropy per particle is different for the superfluid and normal components). Then, we can assume that the total density $n(\mathbf{x}, t)$ and the stream velocity $\mathbf{u}(\mathbf{x}, t)$ obey a simple hydrodynamic equation of motion. The convective derivative of the stream velocity \mathbf{u} is the local acceleration, which depends on the forces arising from the local pressure P and the trap potential U . For irrotational flow, $\nabla \times \mathbf{u} = 0$, we have $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(\mathbf{u}^2/2)$. Then,

$$m \frac{\partial \mathbf{u}}{\partial t} = -\nabla \left(\frac{m}{2} \mathbf{u}^2 + U \right) - \frac{\nabla P}{n}, \quad (7)$$

where m is the atom mass [25].

For a unitary Fermi gas, the local pressure, Eq. (3) takes the general form $P(n, T) = n^{5/3} f_P[T/T_F(n)]$.

Initially, the gas is contained in a harmonic trap at a uniform temperature T_0 and has a density $n_0 \equiv n_0(\tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}}$ is the position in the initial distribution. The initial pressure $P_0(\tilde{\mathbf{x}}) = n_0^{5/3} f_P[T_0/T_F(n_0)]$. Force balance, Eq. (4), requires $\nabla_{\tilde{\mathbf{x}}} P_0(\tilde{\mathbf{x}})/n_0 = -\nabla_{\tilde{\mathbf{x}}} U(\tilde{\mathbf{x}})$.

The hydrodynamic equation of motion can be solved by assuming a scaling ansatz [26,27], where each dimension changes by a scale factor $b_i(t)$, $i = x, y, z$, and $b_i(0) = 1$. The density and stream velocity then take the forms $n(\mathbf{x}, t) = n_0(\tilde{\mathbf{x}})/\Gamma$ and $u_x = x \dot{b}_x(t)/b_x(t)$ and similarly for u_y, u_z . Here $\tilde{\mathbf{x}} \equiv (x/b_x, y/b_y, z/b_z)$ is the position at time $t = 0$ for an atom that is at position \mathbf{x} at time t and $\Gamma \equiv b_x b_y b_z$ is the volume scale factor. The scaling ansatz is exact if the gas is contained in a harmonic trap and the pressure takes the form $P = cn^\gamma$, where c and γ are constants [26,27].

At nonzero temperature, the pressure does not, in general, obey such a simple power law, as the function f_P can be dependent on $T/T_F(n)$ in a complicated way. However, for a gas expanding under isentropic conditions, an exact scaling solution can be obtained, and it predicts temperature-independent expansion. The results correspond closely with our measurements, where nearly temperature-independent breathing frequencies are observed, as we now show.

If the gas is locally isentropic, according to Eq. (2), the local reduced temperature does not change as the gas expands, i.e., $T(\mathbf{x})/T_F[n(\mathbf{x})] = T_0/T_F[n_0(\tilde{\mathbf{x}})]$. Using $n = n_0/\Gamma$ then requires $T(\mathbf{x}) = T_0/\Gamma^{2/3}$. If local equilibrium is maintained, the pressure P is then simply related to P_0 . Using $f_P[T/T_F[n(\mathbf{x})]] = f_P[T_0/T_F[n_0(\tilde{\mathbf{x}})]]$ we must have $P(\mathbf{x}) = P_0(\tilde{\mathbf{x}})/\Gamma^{5/3}$. Then with $x = b_x \tilde{x}$ and $\nabla_{\tilde{\mathbf{x}}} \rightarrow \nabla_{\mathbf{x}}$ in Eq. (4) for $P_0(\tilde{\mathbf{x}})$, we obtain $\nabla P(\mathbf{x})/n(\mathbf{x}) = -\nabla U(\tilde{\mathbf{x}})/\Gamma^{2/3}$. Using this result in Eq. (7) yields

$$m \frac{\partial \mathbf{u}}{\partial t} = -\nabla \left(\frac{m}{2} \mathbf{u}^2 + U(\mathbf{x}) - \frac{U(\tilde{\mathbf{x}})}{\Gamma^{2/3}} \right), \quad (8)$$

where $U(\mathbf{x})$ is the trapping potential while $U(\tilde{\mathbf{x}})$ arises from the pressure force, which is evaluated at $\tilde{x} = x/b_x(t)$ and similarly for \tilde{y}, \tilde{z} . Equation (8) can be obtained for a strongly collisional gas by using a phase-space scaling ansatz [28].

We see that under isentropic conditions, the gas obeys the same hydrodynamic equation at all temperatures as for a zero-temperature gas, where $P = cn^{5/3}$ and c is a constant, even though the pressure and density may assume a more complicated form. For a harmonic potential, all terms in Eq. (8) are linear in x and the scaling solution is exact [26]. Hence, after release from a harmonic trap, where $U(\mathbf{x}) \rightarrow 0$ in Eq. (8), the cloud expands precisely by a scale transformation at all temperatures, similar to that observed in our previous measurements for a strongly interacting Fermi gas at very low temperature [6].

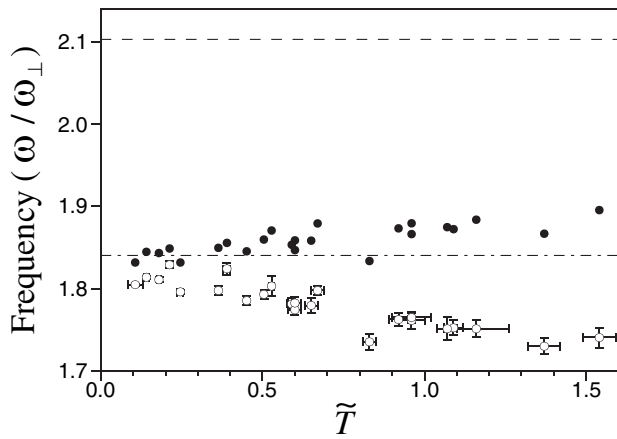


FIG. 2. Radial breathing frequency ω/ω_{\perp} versus empirical temperature \tilde{T} for a unitary gas of ${}^6\text{Li}$ from Ref. [19], showing small variation. Open circles: Measured frequencies. Solid dots: Data after correction for anharmonicity. Lower dot-dashed line: Unitary hydrodynamic frequency, $\omega/\omega_{\perp} = 1.84$. Upper dashed line: Noninteracting gas frequency $2\omega_x/\omega_{\perp} = 2.10$.

The breathing frequencies are readily obtained using $b_i = 1 + \epsilon_i$, where $\epsilon_i \ll 1$. For the radial mode in a cylindrically symmetrical trap, i.e., with $\omega_x = \omega_y = \omega_{\perp} \gg \omega_z$, one obtains $\omega = \sqrt{10/3}\omega_{\perp} = 1.83\omega_{\perp}$. However, for our trap conditions, the exact result is $\omega = 1.84\omega_{\perp}$, independent of the mean field and superfluid contributions that are included in the general form of the energy density.

Figure 2 shows the measured breathing mode frequencies as a function of empirical temperature \tilde{T} , $0.11 \leq \tilde{T} \leq 1.54$. Here, \tilde{T} is determined from the measured spatial profiles as in Ref. [13] and can be calibrated to the theoretical profiles [13], which show that the corresponding T/T_F varies from 0.12 to 1.1. After correction for anharmonicity, the frequency is close to the universal hydrodynamic value over the range of temperatures studied. These results are consistent with nearly isentropic conditions at the highest temperatures, although the system is likely to be changing from a superfluid to a unitary collisional fluid, as suggested by the transition in the damping rate [19]. The observed hydrodynamic behavior at higher temperatures is not explained by existing theories. In particular, the momentum relaxation rate predicted in two-body collision models is much too small to explain the observed hydrodynamic behavior [29].

We are indebted to Jason Ho for a discussion of isentropic hydrodynamics, and to Jason Ho, Kathy Levin, and Qijin Chen for a critical reading of the manuscript. This research is supported by the Physics Divisions of the Army Research Office and the National Science Foundation, the Physics for Exploration program of the National Aeronautics and Space Administration, and the Chemical Sciences, Geosciences and Biosciences Division of the Office of Basic Energy Sciences, Office of Science, U.S. Department of Energy.

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