Breakdown of One-Parameter Scaling in Quantum Critical Scenarios for High-Temperature Copper-Oxide Superconductors

Philip Phillips

Loomis Laboratory of Physics, University of Illinois at Urbana-Champaign, 1110 W.Green St., Urbana, Illinois, 61801-3080, USA

Claudio Chamon

Department of Physics, Boston University, Boston, Massachusetts 02215, USA
(Received 8 December 2004; published 31 August 2005)

We show that if the excitations which become gapless at a quantum critical point also carry the electrical current, then a resistivity linear in temperature, as is observed in the copper-oxide high-temperature superconductors, obtains only if the dynamical exponent z satisfies the unphysical constraint, $z < 0$. At fault here is the universal scaling hypothesis that, at a continuous phase transition, the only relevant length scale is the correlation length. Consequently, either the electrical current in the normal state of the cuprates is carried by degrees of freedom which do not undergo a quantum phase transition, or quantum critical scenarios must forgo this basic scaling hypothesis and demand that more than a single-correlation length scale is necessary to model transport in the cuprates.

The central problem posed by the normal state of the high-temperature copper-oxide superconductors (cuprates) is the riddle of the T-linear resistivity [1]. Namely, over a funnel-shaped region in the temperature-doping plane (as in Fig. 1), the resistivity is a linear function of temperature rather than the T^2 dependence indicative of typical metals. Equally perplexing is the persistence of this transport anomaly to unusually high temperatures, roughly 1000 K. At present, there is no consensus as to the origin of this robust phenomenon. However, two scenarios, (1) marginal Fermi liquid (MFL) phenomenology [2] and (2) quantum criticality [3,4] have been advanced. The former rests on a proposed form for the self-energy that both describes the broad line shapes observed in angularly resolved photoemission and yields a scattering rate, and hence a conductivity, that scales linearly with temperature. In contrast, quantum criticality provides a first-principles account. At the quantum critical coupling, or in the quantum critical regime, the only energy scale governing collisions between quasiparticle excitations of the order parameter is $k_B T$. Consequently, the transport relaxation rate scales as

$$\frac{1}{\tau_{\text{tr}}} \propto \frac{k_B T}{\hbar}, \qquad (1)$$

thereby implying a T-linear resistivity if (naively) the scattering rate is what solely dictates the transport coefficients. While MFL fitting [2] also achieves a scattering rate of this form, a $T = 0$ phase transition is not necessarily the operative cause. The fact that temperature emerges as the only scale in the quantum critical regime regardless of the nature of the quasiparticle interactions is a consequence of universality. Equation (1) is expected to hold as long as the inequalities $T > |\Delta|$ and $t < 1/|\Delta|$ are maintained, with Δ the energy scale measuring the distance from the critical

point and t the observation time. The energy scale $\Delta \propto \delta^{z\nu}$ varies as a function of some tuning parameter $\delta = g - g_c$, where ν is the correlation length exponent and z is the dynamical exponent. At the critical coupling $\delta = 0$ or $g = g_c$, the energy scale Δ vanishes. Ultimately, the observation time constraint, $t < 1/|\Delta|$ implies that only at the quantum critical point is the T-linear scattering rate obtained for all times. That the quantum critical regime is funnel shaped follows from the inequality $T > |\Delta|$. The funnel-shaped critical region should be bounded by a temperature T_{upper} above which the system is controlled by high-energy processes. That quantum criticality is op-
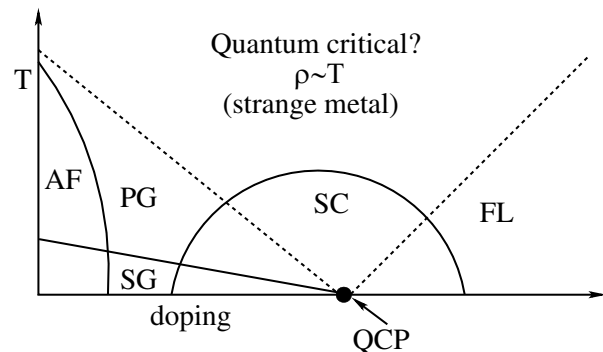

FIG. 1. Heuristic phase diagram of the high-temperature copper-oxide superconductors as a function of temperature and hole concentration (doping). The phases are as follows: AF represents antiferromagnet; SG, the spin glass; and SC, the superconductor. The spin-glass phase terminates at a critical doping level (quantum critical point, QCP) inside the dome. The dashed lines indicate crossovers not critical behavior. In this context, PG is the pseudogap phase in which the single-particle density of states develops a dip at the chemical potential and FL is the Fermi liquid phase.

erative up to temperatures of order $T = 1000$ K in the cuprates is questionable, but we relax this criticism in carrying out our analysis of the scaling of the resistivity.

Because Eq. (1) is valid for any $T = 0$ phase transition, it has been quickly adopted as the explanation of choice for the T-linear resistivity in the cuprates. In fact, there has been no paucity [5–9] of candidate quantum critical points proposed for the cuprates: (1) at $1/8^{th}$-hole doping in $Bi_2Sr_{2-x}La_xCuO_{6+\delta}$ at 58 T, a transition [5] occurs between an anisotropic 2D and a 3D superconductor, (2) near optimal doping, the Hall coefficient [6] exhibits a significant break, indicating a fundamental restructuring of the Fermi surface, and (3) in $La_{2-x}Sr_xCu_{0.95}Zn_{0.05}O_4$ a spin-glass state terminates [8] at roughly optimal doping, $x \approx 0.19$. Regardless of its origin, a $T = 0$ phase transition near optimal doping can, in principle, account for the funnel-shaped T-linear region seen in early transport experiments. However, several experiments [10,11] call into question the very existence of such a wide region. Namely, Raffy, *et al.* [10] and also Ando, *et al.* [11] find that the T-linear region is not a region at all, existing only at optimal doping. While Ando, *et al.* [11] argue that the absence of a triangular region (in the T-x plane) near optimal doping strongly suggests that quantum criticality is not the cause of the T-linear resistivity, an equally valid explanation is that the time constraint $t < 1/|\Delta|$ is violated except at optimal doping. Nonetheless, optical transport measurements [9] find that contrary to theoretical predictions [12], the optical conductivity does not obey the predicted scaling law $T^{-\mu}f(\omega/T)$ with a constant μ as ω/T is varied. In contrast, they find [9] that $\mu = 1$ for $\omega/T < 1.5$ and $\mu \approx 0.5$ for $\omega/T > 3$. In addition, in the quantum critical regime of other strongly correlated systems, such as the heavy fermions [13], the resistivity exhibits a nonuniversal algebraic temperature dependence of the form $\rho \propto \rho_0 + AT^\alpha$ with $0.3 < \alpha < 2.0$ and $A < 0$ or $A > 0$.

Motivated by such experiments, we examine what conditions must hold for T-linear resistivity to be compatible with the universally accepted assumption that, at a continuous quantum critical point, the only relevant length scale is the correlation length. We obtain, using the single scale hypothesis and the fact that electric charge is conserved, a very general scaling law for the electric conductivity near a quantum critical point. This scaling law must hold irrespective of the microscopic details of the theory, and regardless of the quantum statistics of the charge carriers, be they bosons or fermions. From the scaling law, we find that T-linear resistivity is obtained only if the dynamical exponent $z < 0$, which is an unphysical negative value. Consequently, no consistent account of T-linear resistivity is possible if the quantum critical modes carry the electrical charge. We conclude that either the degrees of freedom that are responsible for the T-linear resistivity in the cuprates are not undergoing a quantum phase transition, or that quantum critical scenarios must relinquish the simple single scale hypothesis to explain the resistivity law in the cuprates.

To proceed, we derive a general scaling form for the conductivity near a quantum critical point. Consider a general action S, the microscopic details of which are unimportant. An externally applied electromagnetic vector potential A^μ, $\mu = 0, 1, \ldots, d$, couples to the electrical current, j_μ, so that

$$S \to S + \int d\tau d^d x A^\mu j_\mu. \tag{2}$$

The one-parameter scaling hypothesis in the context of quantum systems is that spatial correlations in a volume smaller than the correlation volume, ξ^d, and temporal correlations on a time scale shorter than $\xi_t \propto \xi^z$ are small, and space-time regions of size $\xi^d \xi_t$ behave as independent blocks. With this hypothesis in mind, we write the scaling form for the singular part of the logarithm of the partition function by counting the number of correlated volumes in the whole system:

$$\ln Z = \frac{L^d \beta}{\xi^d \xi_t} F(\delta \xi^{d_\delta}, \{A^i_\lambda \xi^{d_A}\}). \tag{3}$$

In this expression, L is the system size, $\beta = 1/k_B T$ the inverse temperature, δ the distance from the critical point, and d_δ and d_A the scaling dimensions of the critical coupling and vector potential, respectively. The variables $A^i_\lambda = A^i(\omega = \lambda \xi_t^{-1})$ correspond to the (uniform, $k = 0$) electromagnetic vector potential at the scaled frequency $\lambda = \omega \xi_t$, and $i = 1, \ldots, d$ labels the spatial components. Two derivatives of the logarithm of the partition function with respect to the electromagnetic gauge $A^i(\omega)$,

$$\sigma_{ij}(\omega, T) = \frac{1}{L^d \beta} \frac{1}{\omega} \frac{\delta^2 \ln Z}{\delta A^i(-\omega) \delta A^j(\omega)}$$

$$= \xi^{-d} \frac{\xi_t^{-1}}{\omega} \xi^{2d_A} \frac{\delta^2}{\delta A^i_{-\bar\lambda} \delta A^j_{\bar\lambda}} F(\delta = 0)|_{\substack{\bar\lambda = \omega \xi_t \\ \{A^i_\lambda = 0\}}}$$

$$= \frac{Q^2}{\hbar} \xi^{2d_A - d} \sum_{ij} (\omega \xi_t), \tag{4}$$

determine the conductivity for carriers with charge Q. We have explicitly set $\delta = 0$ as our focus is the quantum critical regime. At finite temperature, the time correlation length is cut off by the temperature as $\xi_t \propto 1/T$, and $\xi_t \propto \xi^z$. The engineering dimension of the electromagnetic gauge is unity ($d_A = 1$). Charge conservation prevents the current operators from acquiring an anomalous dimension, hence, that $d_A = 1$ is exact [14]. We then arrive at the general scaling form

$$\sigma(\omega, T) = \frac{Q^2}{\hbar} T^{(d-2)/z} \Sigma\left(\frac{\hbar\omega}{k_B T}\right) \tag{5}$$

for the conductivity where Σ is an explicit function only of the ratio, ω/T. (We have dropped the ij tensor indices for simplicity.) This scaling form generalizes to finite T and ω the $T = 0$ frequency dependent critical conductivity originally obtained by Wen [14]. The generic scaling form,

Eq. (5), is also in agreement with that proposed by Damle and Sachdev [12] in their extensive study of collision-dominated transport near a quantum critical point (see also the scaling analysis in Ref. [15]). What the current derivation lays plain is that regardless of the underlying statistics or microscopic details of the Hamiltonian, be it bosonic (as in the work of Damle and Sachdev [12]) or otherwise, be it disordered or not, the general scaling form of the conductivity is the same. A simple example where such a scaling formula for the conductivity applies is the Anderson metal-insulator transition in $d = 2 + \epsilon$, which can be thought of as a quantum phase transition where the dimensionless disorder strength is the control parameter [16,17].

In the dc limit,

$$\sigma(\omega = 0) = \frac{Q^2}{\hbar} \Sigma(0) \left(\frac{k_B T}{\hbar c}\right)^{(d-2)/z}. \quad (6)$$

In general $\Sigma(0) \neq 0$ [18]. Else, the conductivity is determined entirely by the nonsingular and hence noncritical part of the free energy. The cuprates are anisotropic 3-dimensional systems. Hence, the relevant dimension for the critical modes is $d = 3$ not $d = 2$. In the latter case, the temperature prefactor is constant. For $d = 3$, we find that T-linear resistivity obtains only if $z = -1$. Such a negative value of z is unphysical as it implies that energy scales diverge for long wavelength fluctuations at the critical point. In fact, that the exponent of the temperature prefactor in Eq. (5) is strictly positive is inconsistent with the Drude formula for the conductivity. Consider the work of van der Marel, *et al.* [9] in which a Drude form for the conductivity,

$$\sigma_{\text{Drude}} = \frac{1}{4\pi} \frac{\omega_{\text{pl}}^2 \tau_{\text{tr}}}{1 + \omega^2 \tau_{\text{tr}}^2}, \quad (7)$$

was used to collapse their optical conductivity to a function of ω/T (ω_{pl} is the plasma frequency). Because $\tau_{\text{tr}} \propto 1/T$, the Drude form for the conductivity is consistent with the critical scaling form for the conductivity, Eq. (5), only if $z = -1$. The presence of another energy scale [19] in the Drude formula, namely, the plasma frequency, is also at odds with the scaling form in Eq. (5). On dimensional grounds, the $z = -1$ result in the context of the Drude formula is a consequence of compensating the square power of the plasma frequency with powers of the temperature so that the scaling form Eq. (5) is maintained. Hence, data collapse according to the Drude formula is not an indication that the universality which underlies the scaling form of Eq. (5) is present.

A further indication that the standard picture of quantum criticality fails for the cuprates is found in the application of Eq. (6) to the universal scaling law of Homes *et al.* [20]. Throughout the entire phase diagram of the cuprates, Homes *et al.* [20] have found the empirical relationship,

$$\rho_s = \sigma_{\text{dc}}(T_c^+)T_c \quad (8)$$

between the superfluid density, ρ_s, the superconducting transition temperature, T_c, and the dc conductivity just above T_c, $\sigma_{\text{dc}}(T_c^+)$, holds within an accuracy of 5%. By using the Drude formula for σ_{dc} and Tanner's [21] empirical relationship between the superfluid and normal state densities, namely, $\rho_s = \rho_N/4$, Zaanen [22] has shown that Homes's Law reduces to Eq. (1). That is, the charge degrees of freedom in high T_c superconductors are at the quantum limit of dissipation, referred to by Zaanen as the Planckian limit. Such Planckian dissipators are necessarily quantum critical. However, the conclusion that Homes's Law represents a simple statement about the quantum limit of dissipation relies on the Drude formula, which, as we have discussed, has nothing to do with quantum criticality. To assess the relevance of quantum criticality to Homes's Law, it is more appropriate to use Eq. (6). Substituting Eq. (6) into Eq. (8) results in a simple algebraic relationship [23],

$$\rho_s \propto T_c^{(d-2)/z+1} \quad (9)$$

between the superfluid density and T_c. Regardless of the exponent, this expression has a maximum whenever T_c is maximized and hence is reminiscent of the Uemura relationship [24], another empirical relationship valid only in the underdoped regime. A key failure of the Uemura relationship is optimal doping where ρ_s and T_c are not simultaneously maximized. Hence, we find that the form of the dc conductivity dictated by quantum criticality fails to capture the physics of Homes's Law, an empirical observation valid regardless of doping. Perhaps some as of yet to be discovered form of quantum criticality can explain Homes's law, but such an explanation must lie outside the one-parameter scaling hypothesis.

The inability of Eq. (5) to lead to a consistent account of T-linear resistivity or Homes's Law [20] in the cuprates leaves us with three options. (1) Either T-linear resistivity is not due to quantum criticality, (2) additional noncritical degrees of freedom are necessarily the charge carriers, or (3) perhaps some new theory of quantum criticality can be constructed in which the single-correlation length hypothesis is relaxed. In a scenario involving noncritical degrees of freedom, fermionic charge carriers in the normal state of the cuprates could couple to a critical bosonic mode. Such an account is similar to that in magnetic systems [3] in which fermions scatter off massless bosonic density or spin fluctuations and lead to an array of algebraic forms for the resistivity [25] ranging from $T^{4/3}$ to $T^{3/2}$ in antiferromagnetic and ferromagnetic systems, respectively. While disorder can alter the exponent [26], T-linear resistivity results only in a restricted parameter space. Consequently, in the context of the cuprates, any explanation of T-linear resistivity based on quantum criticality (as it is currently formulated) must rely on the fortuitous presence of a bosonic mode whose coupling to the fermions remains unchanged up to a temperature of $T = 1000$ K. Currently, no such mode which is strictly bosonic is known. This is not

surprising in light of the fact that numerous experimental systems exist [13] in which T-linear resistivity does not occur in the quantum critical regime or T-linear resistivity exists only at a single point rather than a funnel-shaped region [10,11]. These experiments imply that the correspondence between quantum criticality and T-linear resistivity is not one of necessity.

What about new scenarios [27,28] for quantum critical phenomena? For example, an additional length scale, as is the case in deconfined quantum criticality [28], could provide the flexibility needed to obtain T-linear resistivity while still maintaining $z > 0$. A likely scenario is as follows. Entertain the possibility that an additional length scale $\tilde{\xi}$ is relevant which diverges as $\tilde{\xi} \propto \xi^a$, with $a > 1$. If in the calculation of the correlation volume entering Eq. (3), one replaces ξ^d with $\xi^d \rightarrow \ell^d = \xi^d h(\tilde{\xi}/\xi)$, $h(y) = y^{-\lambda}$ a general scaling function, then one is in essence reducing the effective dimensionality such that $d \rightarrow d^* = d - \lambda(a - 1)$. T-linear resistivity results now if $z = 2 - d^*$. The reduction in the effective dimensionality, $\lambda(a - 1)$, can now be fine tuned so that $d^* \leq 1$, thereby resulting in physically permissible values of the dynamical exponent, $z \geq 1$. Nonetheless, such fine scripting of two length scales is also without basis at this time.

Indeed, it is unclear what remedy is appropriate to square single parameter scaling with T-linear resistivity in the cuprates. It might turn out that quantum criticality is not relevant to the problem. What is clear, however, is that if T-linear resistivity is due to quantum criticality of the degrees of freedom that carry the electrical charge, then a consistent theory must be constructed to account for the breakdown of one-parameter scaling. In fact, recent experiments on La_2CuO_4 [29] find that the exponent of the temperature prefactor of the magnetic susceptibility [29] varies across the critical region. Perhaps this variation provides further evidence that physics beyond the standard paradigms is necessary to explain the magnetic and transport properties of the cuprates.

[1] For a review, see B. Batlogg, H. Takagi, H. L. Kao, and J. Kwo, *Electronic Properties of High-T_c Superconductors*, edited by H. Kuzmany *et al.* (Springer-Verlag, Berlin, 1993) pp. 5–12.

[2] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, and A. E. Ruckenstein, Phys. Rev. Lett. **63**, 1996 (1989); E. Abrahams and C. M. Varma, Proc. Natl. Acad. Sci. U.S.A. **97**, 5714 (2000).

[3] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[4] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B **39**, 2344 (1989).

[5] Y. Ando *et al.*, Phys. Rev. Lett. **92**, 247004 (2004).

[6] F. F. Blakirev *et al.*, Nature (London) **424**, 912 (2003).

[7] T. Valla *et al.*, Science **285**, 2110 (1999).

[8] C. Panagopoulos, A. P. Petrovic, A. D. Hillier, J. L. Tallon, C. A. Scott, and B. D. Rainford, Phys. Rev. B **69**, 144510 (2004).

[9] D. van der Marel *et al.*, Nature (London) **425**, 271 (2003).

[10] Z. Konstantinovic, Z. Z. Li, and H. Raffy, Physica (Amsterdam) **351C**, 163 (2001).

[11] Y. Ando *et al.*, Phys. Rev. Lett. **93**, 267001 (2004).

[12] K. Damle and S. Sachdev, Phys. Rev. B **56**, 8714 (1997).

[13] G. R. Stewart, Rev. Mod. Phys. **73**, 797 (2001).

[14] X.-G. Wen, Phys. Rev. B **46**, 2655 (1992); For the analogous result for relativistic field theories, see D. J. Gross, in *Methods in Field Theory: Les Houches 1975*, edited by R. Balian and J. Zin-Justin (North-Holland, Amsterdam, 1975), p. 181.

[15] T. Schneider and J. M. Singer, *Phase Transition Approach to High Temperature Superconductivity: Universal Properties of Cuprate Superconductors* (Imperial College Press, London, 2000). p. 238.

[16] E. Abrahams *et al.*, Phys. Rev. Lett. **42**, 673 (1979).

[17] F. J. Wegner, Z. Phys. B **35**, 207 (1979).

[18] While it is certainly reasonable to assume that $\Sigma(0)$ is finite at zero temperature, we cannot rule out the possibility that $\Sigma(0)$ might in fact diverge. A possible scenario for how this state of affairs might obtain is that a dangerously irrelevant operator could control the conductivity. In this case, $\Sigma(0) \sim 1/T^p$, and T-linear behavior is obtained if $p = (d - 2)/z + 1$. This can only occur above the upper critical dimension. In this regime, all criticality is described by mean-field theory. Hence, a possibility that cannot be eliminated at this time is that all criticality in the cuprates is inherently mean-field and dangerously irrelevant operators control the conductivity. For low-dimensional strongly correlated systems, however, it is unlikely that criticality is inherently mean field.

[19] In systems with a Fermi surface, it is the Fermi energy that replaces the plasma frequency in Eq. (7). Transport in such systems is also incompatible with one-parameter scaling.

[20] C. C. Homes *et al.*, Nature (London) **430**, 539 (2004).

[21] D. R. Tanner *et al.*, Physica (Amsterdam) **244B**, 1 (1998).

[22] J. Zaanen, Nature (London) **430**, 512 (2004).

[23] At the ends of the superconducting dome, a similar algebraic relationship (though with different exponents) describes the zero-temperature superfluid density's dependence on T_c. See A. Kopp and S. Chakravarty, cond-mat/0503298.

[24] Y. J. Uemura, Phys. Rev. Lett. **62**, 2317 (1989).

[25] T. Moriya and T. Takimoto, J. Phys. Soc. Jpn. **65**, 3294 (1996).

[26] A. Rosch, Phys. Rev. Lett. **82**, 4280 (1999).

[27] Q. M. Si *et al.*, Nature (London) **413**, 804 (2001).

[28] T. Senthil *et al.*, Science **303**, 1490 (2004).

[29] Y. Chen *et al.*, cond-mat/0408547.