

Three-Loop Mixing of Dipole Operators

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(Received 5 May 2005; published 2 September 2005)

We calculate the complete three-loop $\mathcal{O}(\alpha_s^3)$ anomalous dimension matrix for the dimension-five dipole operators that arise in the standard model after integrating out the top quark and the heavy electroweak bosons. Our computation completes the three-loop anomalous dimension matrix of operators that govern low-energy $|\Delta F| = 1$ flavor-changing processes, and represents an important ingredient of the next-to-next-to-leading order QCD analysis of the $\bar{B} \rightarrow X_s \gamma$ decay.

DOI: 10.1103/PhysRevLett.95.102004

PACS numbers: 12.38.Bx, 13.20.He, 13.25.Hw, 13.40.Em

Weak interaction phenomena at energies much below the electroweak scale are most conveniently described in the framework of an effective theory that is derived from the standard model (SM) by integrating out the top quark and the heavy electroweak bosons. The Lagrangian of such an effective theory,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD} \times \text{QED}} + \mathcal{L}_{\text{weak}}, \quad (1)$$

is a sum of the conventional QCD \times QED Lagrangian for the remaining SM fields and a linear combination

$$\mathcal{L}_{\text{weak}} \propto \sum_i C_i(\mu) Q_i, \quad (2)$$

of dimension ≥ 5 operators Q_i that are built out of those light fields. $C_i(\mu)$ are the corresponding Wilson coefficients that depend on the renormalization scale μ .

For most phenomenological applications, only operators of dimensions five and six are relevant. The complete set of such operators consists of: (i) dimension-six four-fermion operators, (ii) dimension-six purely gluonic operators, and (iii) dimension-five dipole operators

$$\bar{\psi} \sigma^{\mu\nu} \psi' F_{\mu\nu}, \quad \text{and} \quad \bar{\psi} \sigma^{\mu\nu} T^a \psi' G_{\mu\nu}^a. \quad (3)$$

Here $\sigma^{\mu\nu} = i/2[\gamma^\mu, \gamma^\nu]$, while ψ and ψ' stand for fermion fields of opposite chiralities. Their flavors may or may not be the same. The electromagnetic and strong field strength tensors are denoted by $F_{\mu\nu}$ and $G_{\mu\nu}^a$, respectively. T^a are the $SU(3)_C$ generators for the considered fermions. For off-shell calculations, additional operators that vanish by the QCD \times QED equations of motion (EOM) must be included (see below).

The structure of $\mathcal{L}_{\text{weak}}$ remains the same in any $SU(3)_C \times U(1)_{\text{em}}$ gauge-invariant extension of the SM that does not contain exotic light bosons. New physics effects are therefore encoded in the values of the Wilson coefficients only.

The dipole operators introduced in Eq. (3) are relevant in a variety of phenomenological applications, ranging from electric and magnetic moments of the leptons and nucleons to radiative decays, such as $\mu \rightarrow e\gamma$, $\Omega \rightarrow \Xi\gamma$, $B \rightarrow K^*\gamma$, $B \rightarrow \rho\gamma$, and $\bar{B} \rightarrow X_s\gamma$.

The Wilson coefficients are determined by matching Green's functions of the effective theory and the SM (or its extension) at the electroweak (or higher) scale μ_{high} . Next, one applies the renormalization group equations

$$\mu \frac{d}{d\mu} C_i(\mu) = \gamma_{ji}(\mu) C_j(\mu), \quad (4)$$

to evolve $C_i(\mu)$ to the relevant low-energy scale μ_{low} . In this way, large logarithms $\ln(\mu_{\text{high}}^2/\mu_{\text{low}}^2)$ are resummed from all orders of the perturbation series.

Neglecting QED effects, the anomalous dimension matrix (ADM) $\hat{\gamma}(\mu)$ has the following perturbative expansion:

$$\hat{\gamma}(\mu) = \sum_{k=0} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{k+1} \hat{\gamma}^{(k)}, \quad (5)$$

where $\alpha_s = g_s^2/(4\pi)$ is the strong coupling constant.

The purpose of our present work is to evaluate the entries of $\hat{\gamma}^{(2)}$ that correspond to the $\mathcal{O}(\alpha_s^3)$ strong mixing of the dipole operators of Eq. (3) containing quark fields. These entries can be extracted from the three-loop QCD renormalization constants in the effective theory. Lower-order entries are already known from previous calculations [1–5]. The three-loop QCD self-mixing of the electromagnetic dipole operator coincides with the anomalous dimension of the rank-two antisymmetric tensor current that has been calculated in Ref. [6]. We confirm all these findings. Our remaining three-loop results are entirely new.

The main phenomenological motivation for our work is to provide a new contribution to the calculation of the $\bar{B} \rightarrow X_s \gamma$ branching ratio at the next-to-next-to-leading order (NNLO) in QCD. Including these $\mathcal{O}(\alpha_s^2)$ corrections is necessary to reduce the theoretical uncertainty of the SM calculation [7] below the current experimental one [8]. Several steps in this direction have already been made [9–11]. Our findings can also be relevant for the CP -odd electric dipole moment of the neutron, provided the new physics matching scale is sufficiently high.

Our calculation completes the three-loop QCD ADM for the whole dimension-five part of the effective Lagrangian, because the dipole operators in Eq. (3) are the only EOM-

non-vanishing operators in this sector. Simultaneously, our results establish the whole three-loop QCD ADM for the $|\Delta F| = 1$ operators of dimensions five and six that arise in the SM case—all other three-loop ADM entries for such operators are known from previous publications [5,11,12]. The only ADM entries that remain to be calculated for $\bar{B} \rightarrow X_s \gamma$ at the NNLO in QCD correspond to the four-loop mixing of certain four-quark operators into the dipole operators [13].

For definiteness in the further discussion, we shall choose the flavors, chiralities, normalization and names of the dipole operators as it is usually done in the phenomenological analyses of $\bar{B} \rightarrow X_s \gamma$, namely

$$\begin{aligned} Q_7 &= \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}, \\ Q_8 &= \frac{g_s}{16\pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a. \end{aligned} \quad (6)$$

However, we stress that the 2×2 ADM which we calculate is the same for any pair of such quark dipole operators, including the flavor-conserving ones. There is no mixing between dipole operators of different flavor content, even in the flavor-conserving sector [3,6].

In order to remove the divergences of all possible off-shell one-particle-irreducible Green's functions with single insertions of Q_7 and Q_8 , we have to introduce the following EOM-vanishing counterterms [3,5]

$$\begin{aligned} Q_{\not{D}\not{D}} &= \frac{1}{16\pi^2} m_b \bar{s}_L \not{D}\not{D} b_R, \\ Q_{\not{D}\not{G}} &= \frac{ig_s}{16\pi^2} m_b \bar{s}_L (\not{D}\not{G} - \not{G}\not{D}) b_R, \end{aligned} \quad (7)$$

where $D_\mu = \partial_\mu + ig_s G_\mu + ieQ_d A_\mu$ and $\tilde{D}_\mu = \tilde{\partial}_\mu - ig_s G_\mu - ieQ_d A_\mu$ denote the covariant derivatives of the gauge group $SU(3)_C \times U(1)_{\text{em}}$ acting on the fields to the right and left, respectively, and we have used the definition $G_\mu = G_\mu^a T^a$ for the matrix-valued gluon field. A_μ is the photon field, and the color generators are normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}/2$.

Notice that the operator $Q_{\not{D}\not{D}}$ is gauge invariant, while $Q_{\not{D}\not{G}}$ is not. The appearance of such operators is expected on general grounds [14,15]. In principle, one could also encounter nonphysical counterterms that can be written as Becchi-Rouet-Stora-Tyutin variations of some other operators, so-called BRST-exact operators. However, they turn out to be unnecessary in the case of the dipole operator mixing. This issue is discussed in more detail in Refs. [3,5,11].

We perform the calculation using dimensional regularization and the $\overline{\text{MS}}$ scheme. As far as the matrix γ_5 is concerned, its only relevant property in our case is $[\gamma_5, \gamma_\mu \gamma_\nu] = 0$, which, to our knowledge, holds in all the commonly used schemes for the treatment of γ_5 , including the naive dimensional regularization and t'Hooft-Veltman schemes. However, the ADM beyond one loop in the dimensional reduction scheme is different from the one we find here because this scheme does not

coincide with standard dimensional regularization even in the absence of γ_5 . A description of the properties of the naive dimensional regularization, t'Hooft-Veltman and dimensional reduction schemes, as well as a list of relevant original articles can be found in Ref. [16].

The necessary three-loop renormalization matrix is found by calculating the one- and two-loop $b \rightarrow s$, $b \rightarrow s\gamma$, $b \rightarrow sg$ amputated Green's functions with single insertions of Q_7 , Q_8 , $Q_{\not{D}\not{D}}$, and $Q_{\not{D}\not{G}}$, as well as the three-loop $b \rightarrow s\gamma$ and $b \rightarrow sg$ amplitudes with insertions of Q_7 and Q_8 . Sample diagrams are shown in Fig. 1. The corresponding one-, two- and three-loop amplitudes are evaluated using the method that has been described in Refs. [3,5,17]. We perform the calculation off shell in an arbitrary R_ξ gauge, which allows us to explicitly check the gauge-parameter independence of the mixing among physical operators. To distinguish between infrared and ultraviolet (UV) divergences, we introduce a common mass M for all fields, expanding all loop integrals in inverse powers of M . This makes the calculation of the UV divergences possible at three loops, as M becomes the only relevant internal scale, and three-loop tadpole integrals with a single nonzero mass are known [17,18]. On the other hand, this procedure requires taking into account insertions of the nonphysical operators $Q_{\not{D}\not{D}}$ and $Q_{\not{D}\not{G}}$, as well as of the following counterterm of dimension three:

$$M^2 m_b \bar{s}_L b_R. \quad (8)$$

A comprehensive discussion of the technical details of the renormalization of the effective theory and the actual calculation of the operator mixing is given in Refs. [3,5].

Having summarized our method, we now present our results for arbitrary numbers of down- and up-type quark flavors denoted by n_d and n_u , respectively. The ADM depends on the total number of active quark flavors $n_f = n_u + n_d$, and their "total" electric charge $\bar{Q} = n_u Q_u + n_d Q_d$. The regularization- and renormalization-scheme independent matrix $\hat{\gamma}^{(0)}$ is given by

$$\hat{\gamma}^{(0)} = \begin{pmatrix} \frac{32}{3} & 0 \\ \frac{32}{3} Q_d & \frac{28}{3} \end{pmatrix}. \quad (9)$$

While the matrix $\hat{\gamma}^{(0)}$ is renormalization-scheme independent, $\hat{\gamma}^{(1)}$ and $\hat{\gamma}^{(2)}$ are not. In the $\overline{\text{MS}}$ scheme, we obtain

$$\hat{\gamma}^{(1)} = \begin{pmatrix} \frac{1936}{9} - \frac{224}{27} n_f & 0 \\ (\frac{368}{3} - \frac{224}{27} n_f) Q_d & \frac{1456}{9} - \frac{61}{27} n_f \end{pmatrix}, \quad (10)$$

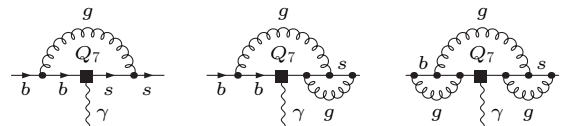


FIG. 1. Typical examples of one-particle-irreducible diagrams describing the self-mixing of Q_7 at the one-, two- and three-loop level.

and

$$\hat{\gamma}^{(2)} = \begin{pmatrix} -\frac{1856}{27} - \frac{1280}{9}n_f & 0 \\ \frac{640}{9}\bar{Q} - (\frac{1856}{27} + \frac{1280}{9}n_f)Q_d & -\frac{28624}{27} - \frac{1312}{9}n_f \end{pmatrix} \zeta(3) \\ + \begin{pmatrix} \frac{307448}{81} - \frac{23776}{81}n_f - \frac{352}{81}n_f^2 & 0 \\ -\frac{1600}{27}\bar{Q} + (\frac{159872}{81} - \frac{17108}{81}n_f - \frac{352}{81}n_f^2)Q_d & \frac{268807}{81} - \frac{4343}{27}n_f - \frac{461}{81}n_f^2 \end{pmatrix}. \quad (11)$$

We remark that the explicit electric charge Q_d originates solely from the quarks in the operators, and thus has to be replaced by Q_u for operators containing up-quark fields. As it is characteristic for three-loop anomalous dimensions, the entries of $\hat{\gamma}^{(2)}$ involve terms proportional to the Riemann zeta function $\zeta(3) \approx 1.20206$.

Of course, the presence of the bottom quark mass in the normalization of the dipole operators of Eq. (6) affects the values of $\hat{\gamma}^{(k)}$. Had we decided to define the operators without quark mass in their normalization, the results in Eqs. (9)–(11) would need to be replaced by $\gamma_{ij}^{(k)} - \gamma_m^{(k)} \delta_{ij}$, where

$$\gamma_m^{(0)} = 8, \quad \gamma_m^{(1)} = \frac{404}{3} - \frac{40}{9}n_f, \\ \gamma_m^{(2)} = 2498 - \left(\frac{4432}{27} + \frac{320}{3}\zeta(3) \right) n_f - \frac{280}{81}n_f^2 \quad (12)$$

are the expansion coefficients of the quark mass anomalous dimension. In particular, verifying that our result for $\gamma_{77}^{(2)}$ is in agreement with Eq. (8) of Ref. [6] requires performing such a replacement because no quark mass is present in the normalization of the tensor current considered there. We

$$\Delta^{(2)}C_7(\mu_b) = \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 \left[\eta^{62/23} C_7^{(2)}(\mu_W) + \frac{8}{3}(\eta^{60/23} - \eta^{62/23})C_8^{(2)}(\mu_W) - \frac{37208}{4761}(\eta^{39/23} - \eta^{62/23})C_7^{(1)}(\mu_W) \right. \\ \left. - \left(\frac{7164416}{357075}\eta^{37/23} - \frac{297664}{14283}\eta^{39/23} - \frac{256868}{14283}\eta^{60/23} + \frac{6698884}{357075}\eta^{62/23} \right) C_8^{(1)}(\mu_W) \right. \\ \left. + (16.6516\eta^{16/23} - 61.0768\eta^{39/23} + 44.4252\eta^{62/23})C_7^{(0)}(\mu_W) + (36.4636\eta^{14/23} - 44.4043\eta^{16/23} \right. \\ \left. - 135.3141\eta^{37/23} + 146.6159\eta^{39/23} + 15.4051\eta^{60/23} - 18.7662\eta^{62/23})C_8^{(0)}(\mu_W) \right], \quad (16)$$

and

$$\Delta^{(2)}C_8(\mu_b) = \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 \left[\eta^{60/23} C_8^{(2)}(\mu_W) - \frac{64217}{9522}(\eta^{37/23} - \eta^{60/23})C_8^{(1)}(\mu_W) + (39.7055\eta^{14/23} \right. \\ \left. - 45.4824\eta^{37/23} + 5.7769\eta^{60/23})C_8^{(0)}(\mu_W) \right]. \quad (17)$$

Explicit expressions for the relevant $b \rightarrow s\gamma$ and $b \rightarrow sg$ matching conditions

$$C_i(\mu_W) = \sum_{k=0} \left(\frac{\alpha_s(\mu_W)}{4\pi} \right)^k C_i^{(k)}(\mu_W), \quad (18)$$

can be found in Ref. [10].

note that there is a misprint in the last line of Eq. (7) of the latter paper. Obviously, the factor C_F^2 should read N_f^2 .

The renormalization group equation for the Wilson coefficients given in Eq. (4) has the following general solution

$$C_i(\mu_b) = U_{ij}(\mu_b, \mu_W) C_j(\mu_W), \quad (13)$$

where the matching and the low-energy scales have been denoted by μ_W and μ_b , respectively. In the $\bar{B} \rightarrow X_s \gamma$ case, one has $\mu_W = \mathcal{O}(M_W)$ and $\mu_b = \mathcal{O}(m_b)$. The evolution matrix $U_{ij}(\mu_b, \mu_W)$ depends on the strong gauge coupling ratio $\eta = \alpha_s(\mu_W)/\alpha_s(\mu_b)$. The ADM that we have calculated allows us to find the complete $\mathcal{O}(\alpha_s^2)$ contributions to

$$\Delta C_7(\mu_b) = \sum_{j=7,8} U_{7j}(\mu_b, \mu_W) C_j(\mu_W), \quad (14)$$

and

$$\Delta C_8(\mu_b) = U_{88}(\mu_b, \mu_W) C_8(\mu_W). \quad (15)$$

Denoting these $\mathcal{O}(\alpha_s^2)$ contributions by $\Delta^{(2)}C_7(\mu_b)$ and $\Delta^{(2)}C_8(\mu_b)$, respectively, and using the general NNLO formalism presented in Refs. [11,19], we obtain for $n_f = 5$, $Q_d = -1/3$, and $\bar{Q} = 1/3$:

Setting $\alpha_s(M_Z) = 0.118$, $m_t = m_t(m_t) = 168.5$ GeV, $\mu_W = M_W = 80.425$ GeV and $\mu_b = 4.8$ GeV, one obtains

$$\Delta^{(2)}C_7(\mu_b) \approx \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 17.2 \approx 0.0051, \quad (19)$$

and

$$\Delta^{(2)}C_8(\mu_b) \approx \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2 6.4 \approx 0.0019. \quad (20)$$

The correction in Eq. (19) causes a suppression of the $\bar{B} \rightarrow X_s \gamma$ branching ratio by around 3%. However, one should bear in mind that the size of the correction depends strongly on the exact value chosen for μ_W . Only the *total* values of $C_i(\mu_b)$ are guaranteed to become less μ_W -dependent once more orders in their perturbative expansion are being included. At the moment, no complete $\mathcal{O}(\alpha_s^2)$ expression for $C_7(\mu_b)$ is available because one and only one element on the right hand side of Eq. (13) remains unknown at this order, namely, the element $U_{72}(\mu_b, \mu_W)$ of the evolution matrix that corresponds to the current-current operator $Q_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$. It is going to become available once the calculation of the four-loop mixing of the relevant four-quark into the dipole operators is completed [13].

To conclude: We have evaluated the complete three-loop $\mathcal{O}(\alpha_s^3)$ mixing among the dipole operators, that is, in the whole dimension-five sector of the effective theory that describes processes occurring much below the electroweak scale. Our results are particularly relevant for the NNLO analysis of radiative B decays in the SM and many of its extensions.

We would like to thank Matthias Steinhauser for providing us with an updated version of MATAD [20]. Our calculations made extensive use of the Fermilab General-Purpose Computing Farms [21]. M. G. and U. H. appreciate the warm hospitality of the Institute T31 for Theoretical Elementary Particle Physics at the Technical University of Munich where part of this work has been performed. The work of U. H. is supported by the U.S. Department of Energy under Contract No. DE-AC02-76CH03000. M. M. was supported in part by the Polish Committee for Scientific Research under the Grant No. 2 P03B 078 26, and from the European Community's Human Potential Programme under the contract no. HPRN-CT-2002-00311, EURIDICE.

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