Nonperturbative Fixed Point in a Nonequilibrium Phase Transition

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We apply the nonperturbative renormalization group method to a class of out-of-equilibrium phase transitions (usually called ''parity-conserving'' or, more properly, ''generalized voter'' class) which is out of the reach of perturbative approaches. We show the existence of a genuinely nonperturbative fixed point, i.e., a critical point that does not seem to be Gaussian in any dimension.

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Our understanding of equilibrium phase transitions is largely due to the success of perturbative renormalization group (RG) methods performed around some (upper or lower) critical dimension d_c and to the existence of integrability and conformal symmetry properties in $d = 2$ [1]. The situation is far less satisfactory out of equilibrium, where the relevant ingredients determining universality classes are sometimes not even known [2,3]. There are a number of technical reasons for this: (i) many systems possess neither a lower d_c nor a low-dimensional exact solution; (ii) even at the critical point, models with a Langevin-like dynamics, that is, those that involve only one time derivative in their kinetic terms, cannot be conformal invariant; (iii) contrary to equilibrium, no RG calculation is, in general, available at and above three-loop order, and this prevents the use of series resummation techniques to compute accurately universal quantities in low dimensions. If, moreover, one keeps in mind that features not accessible to perturbative RG methods may play a crucial role, then the so-called nonperturbative renormalization group (NPRG) approach appears as a method of choice out of equilibrium.

Such nonperturbative effects were evidenced recently in a study of the classic reaction-diffusion problem where particles A diffuse, branch ($A \to 2A$), and annihilate ($2A \to \varnothing$) with rates D, σ, and λ [4]. Whereas, in an important work [5], Cardy and Täuber had shown that perturbative RG calculations led to conclude that no finite-σ transition to the empty absorbing state is possible for $d > 2$, an NPRG study at the nonuniversal level showed that such absorbing phase transitions exist in any finite dimension (with their critical properties in the directed percolation class, as expected from perturbative RG).

In this Letter, we apply the NPRG method to a similar class of absorbing phase transitions, one for which even the *universal* properties are out of the reach of perturbative approaches. We put forward the existence of a genuinely nonperturbative fixed point, i.e., a critical point that does not seem to be Gaussian in any dimension. Our calculations unveil the structure of the RG flow, reveal clearly why perturbative methods are doomed to failure in this case, and provide estimates of critical exponents heretofore accessible only via numerical simulations or series expansions.

We actually treat two classes of systems known to be equivalent in $d = 1$; the physical dimension where the nonperturbative fixed point alluded to above is relevant. The first group [6,7] includes the reaction-diffusion system described above but where the branching reaction now creates pairs of particles ($A \to 3A$), so that, incidently, the parity of the total number of particles is conserved. Improperly named ''parity-conserving'' (PC) class (as argued in [8], where this conservation law was shown to have no influence on similar reaction-diffusion systems), it is best characterized by the second group of problems [9], that of phase transitions into one out of two Z_2-symmetric absorbing states [10]. In $d = 1$, the particles of the PC model can, indeed, be seen as interfaces between ''+'' and ''−'' domains [11]. In this ''spin'' language, domains evolve and compete under Z_2-symmetric rules with noise acting only at interfaces, the definition of the ''generalized voter'' (GV) class as given in [12].

The reaction-diffusion problem ($A \to 3A$, $2A \to \varnothing$) was also studied in [5]. Using the Doi-Peliti formalism [13], one can obtain the following microscopic action [5,6]:

$$S[\phi, \bar{\phi}] = \int_{x,t} \bar{\phi}(\partial_t - D\nabla^2)\phi - \lambda(1 - \bar{\phi}^2)\phi^2 + \sigma(1 - \bar{\phi}^2)\phi\bar{\phi} \quad (1)$$

in terms of the ''physical'' density field ϕ and the associated response field $\bar{\phi}$. Cardy and Täuber first performed an expansion around the upper critical dimension $d_c = 2$ where the transition occurs at zero branching rate σ, so that the fixed point then is that of the pure annihilation problem $2A \to \varnothing$. They showed that this annihilation fixed point remains relevant down to $d = \frac{4}{3}$ (at one-loop order), where it becomes also attractive in the σ direction. In another expansion, this time performed directly in $d = 1$, they were able to identify an appropriate combination of the coupling constants λ, σ, which does admit a fixed point for $d \leq \frac{4}{3}$, although the flow diagram with respect to these

original variables is rather peculiar. Even if their results do suggest that a phase transition should exist at $\sigma \neq 0$—as observed in numerical simulations and mean-field-based expansions of many microscopic models in $d = 1$ [7,9]—the critical exponents remain poorly determined and, worse, the very possibility of computations beyond one-loop order appears to be problematic [5].

As for the GV class, the following Langevin equation was recently proposed [11]:

$$\partial_t \phi = (-\sigma\phi + \mu\phi^3)(1 - \phi^2) + D\nabla^2\phi + \sqrt{2\lambda(1 - \phi^2)}\eta \tag{2}$$

with $\phi \in [-1, 1]$ and η is a delta-correlated Gaussian noise of unit variance. The $(1 - \phi^2)$ factors, appearing both in the deterministic force and in the noise amplitude, impose $\phi = \pm 1$ to be symmetric absorbing states. In $d = 1$, only one type of transition is observed by varying σ for any value of μ, and its critical properties are, indeed, those observed for the so-called parity-conserving models [14]. Taking $\mu = 0$ for the sake of simplicity, the generating functional associated with the simplified Langevin equation is nothing but (1) where $\phi \leftrightarrow \bar{\phi}$ and $t \to -t$. Thus, at this bare level, the two problems are strictly equivalent in $d = 1$. We see later that this equivalence is preserved under the RG flow.

We cannot detail here the implementation of the NPRG but mention only the essential features of the method [15,16]. The main idea is to build a one-parameter family of models, indexed by a momentum scale k, interpolating smoothly between the short-distance physics at the (microscopic) scale $k = \Lambda$, where no fluctuation has been taken into account, and the long-distance physics at scale $k = 0$, where all fluctuations have been integrated out. In Wilson's original formulation, this leads to a flow of effective Hamiltonians—for the slow modes—defined at scale k. Following [15], we focus on the flow of "free energies" for the rapid modes, $|q| \in [k, \Lambda]$, i.e., those already integrated out at this scale. This is achieved by adding a masslike term of order k^2 to the slow modes ($|q| < k$), which "freezes" them. This mass term reads

$$\Delta S_k[\phi, \bar{\phi}] = \int_{q,\omega} R_k(q^2)\bar{\phi}(-q, -\omega)\phi(q, \omega), \tag{3}$$

where a convenient choice [17] of "cutoff" function is

$$R_k(q^2) = k^2(1 - q^2/k^2)\theta(1 - q^2/k^2).$$

The "partition functions" $Z_k[J, \bar{J}] = \int D\phi D\bar{\phi} \exp(-S - \Delta S_k + \int J\phi + \int \bar{J}\bar{\phi})$ become therefore k dependent. Through the Legendre transform of $\log Z_k[J, \bar{J}]$, one obtains the state function Γ_k—analogous to the Gibbs free energy at equilibrium—which depends on the fields $\psi = \delta \log Z_k/\delta J$ and $\bar{\psi} = \delta \log Z_k/\delta \bar{J}$:

$$\Gamma_k[\psi, \bar{\psi}] + \log Z_k[J, \bar{J}] = \int J\psi + \int \bar{J}\bar{\psi} - \int R_k\psi\bar{\psi}. \tag{4}$$

Note that the last term in Eq. (4), proportional to R_k, ensures that Γ_k has the proper limit at $k = \Lambda$: $\Gamma_{k=\Lambda} \sim S$ [15]. The following exact functional differential equation governs the RG flow of Γ_k under an infinitesimal change of the scale $s = \log(k/\Lambda)$ [15,18]:

$$\partial_s \Gamma_k = \frac{1}{2} \operatorname{Tr} \int_{q,\omega} \partial_s \hat{R}_k (\hat{\Gamma}_k^{(2)} + \hat{R}_k)^{-1}, \tag{5}$$

where $\hat{R}_k$ is the symmetric, off-diagonal, 2×2 matrix of element R_k and $\hat{\Gamma}_k^{(2)}[\psi, \bar{\psi}]$ the 2×2 matrix of second derivatives of Γ_k with respect to ψ and $\bar{\psi}$. Obviously, Eq. (5) cannot be solved exactly and one usually truncates it. A standard truncation is the derivative expansion [15] in which Γ_k is expanded as a power series in ∇ and ∂_t. The local potential approximation (LPA), which is the simplest such truncation, consists in keeping only a potential term in Γ_k while neglecting any field renormalization (D is k independent and could be scaled away):

$$\Gamma_k^{\mathrm{LPA}} = \int_{x,t} \{U_k(\psi, \bar{\psi}) + \bar{\psi}(\partial_t - D\nabla^2)\psi\}. \tag{6}$$

If the anomalous dimensions are not too large, the LPA already provides a good description of the effective potential as well as a rather accurate estimate of the exponent ν governing the divergence of the correlation length. Since our main goal is to identify the nonperturbative fixed point governing the PC/GV transition in $d = 1$, we restrict ourselves, in what follows, to the LPA [19].

The NPRG equation for the effective potential, valid for all reaction-diffusion processes involving a single species, has been established in [18]. Studying a particular model amounts to solving this equation in a subspace defined by the symmetries of the problem, starting with the corresponding microscopic action S. The flow equation for the dimensionless potential $u = k^{d+2}U_k$, expressed in terms of the dimensionless fields $\psi \to k^{-d}\psi$ and $\bar{\psi} \to \bar{\psi}$, reads (to lighten notations, we omit the implicit dependence of u on the running scale):

$$\partial_s u = -(d + 2)u + d\psi u^{(1,0)} + V_d\left[1 - \frac{u^{(2,0)}u^{(0,2)}}{(1 + u^{(1,1)})^2}\right]^{-1/2}, \tag{7}$$

where $u^{(n,p)} = \frac{\partial^{n+p}u}{\partial^n\psi\partial^p\bar{\psi}}$ and $V_d = \frac{2^{-d+1}\pi^{-d/2}}{d\Gamma(d/2)}$. In our problem, the effective potential must remain unchanged under the simultaneous transformations $\psi \to -\psi$ and $\bar{\psi} \to -\bar{\psi}$ ("-parity-conservation"/Z_2 symmetry of the PC/GV models) [5,20]. This leads to the existence of three quadratic invariant quantities, ψ^2, $\bar{\psi}^2$, and $\psi\bar{\psi}$, from which all other invariant combinations of the fields can be built. Action (1) can be expressed in terms of these invariants, but it also possesses additional features: the potential of the microscopic action S is proportional to $1 - \bar{\psi}^2$ and vanishes for $\psi = 0$ (in the PC language). One can check that this is also true for $\partial_s u$ once it is for u so that this structure is preserved by the renormalization flow. This further con-

strains the functional subspace in which the running potential evolves. To summarize, the structure of the running potential defining our problem is

$$u(\psi, \bar{\psi}) = (1 - \bar{\psi}^2)\mathcal{F}(\psi^2, \psi\bar{\psi}) \quad \text{with } \mathcal{F}(0,0) = 0. \quad (8)$$

Postponing the numerical resolution of the partial differential Eq. (7) in the functional subspace defined above, we now perform a Taylor series expansion of the potential. In the absence of any information about the radius of convergence, the point around which the expansion is performed matters, all the more so since we want to eventually truncate it. Here, we expand the potential around a non-negative solution of the stationary equations of motion $\frac{\partial u}{\partial \psi} = 0$ and $\frac{\partial u}{\partial \bar{\psi}} = 0$. The former is always satisfied by $\bar{\psi} = 1$, while the solutions of the latter then correspond either to $\psi = 0$ (the "origin") or to a (running) $\psi > 0$ ("the minimum"). We present results obtained around the origin, which involve lighter equations than those obtained around the minimum.

The simplest truncation, which we now analyze in some detail, consists in keeping in u only the two coupling constants already present in S. Inserting this ansatz in Eq. (7), we obtain the following nontrivial flows for the running constants [21]:

$$\partial_s \lambda = -\lambda(2 - d) + 2V_d \frac{\lambda^2(1 + 22\sigma)}{(1 - 2\sigma)^3}, \quad (9)$$

$$\partial_s \sigma = -2\sigma + 6V_d \frac{\lambda\sigma}{(1 - 2\sigma)^2}. \quad (10)$$

This RG flow possesses three fixed points: the trivial, Gaussian, fixed point F_{G} ($\lambda^*_{\mathrm{G}} = \sigma^*_{\mathrm{G}} = 0$), the annihilation fixed point F_{A} ($\lambda^*_{\mathrm{A}} = \frac{2-d}{2V_d}$, $\sigma^*_{\mathrm{A}} = 0$), and the nontrivial fixed point F^* of coordinates:

$$\lambda^* = \frac{192}{V_d(28 - 3d)^2}, \qquad \sigma^* = \frac{4 - 3d}{56 - 6d}. \quad (11)$$

The Gaussian fixed point F_{G} of eigenvalues $(2, 2 - d)$ is relevant above $d_{\mathrm{c}} = 2$, where it coincides with F_{A}. For $d \in [\frac{4}{3}, 2]$, F_{A}, whose eigenvalues are $(d - 2, 3d - 4)$, is relevant. At $d = \frac{4}{3}$, F_{A} and the nontrivial fixed point F^* coincide and exchange stability, so that F^* is the relevant fixed point for $d < \frac{4}{3}$ (Fig. 1). Note that then $\sigma^* > 0$, and thus F^* is in the physical region of parameter space, whereas it plays no role for the physics of reaction-diffusion systems when $d > \frac{4}{3}$. Note also that F^* is not Gaussian in any dimension (at least at this order), and is thus out of the reach of any perturbative expansion. The PC/GV fixed point in $d = 1$ is thus F^*, a genuinely nonperturbative fixed point. Its associated critical exponent, given by the inverse of its negative eigenvalue is $\nu = \frac{12}{\sqrt{149} - 7} \simeq 2.30$. The flow diagram in this dimension is shown in Fig. 2. The once unstable manifold of F^*, connected to F_{G}, is the critical "surface" separating the absorbing and the active phases. The flow around F_{A} is rather peculiar: as d is decreased from $\frac{4}{3}$, the eigenvector of F_{A} that is not parallel to the λ axis rotates and becomes parallel, in $d = 1$, to this axis. Thus, within the LPA, F_{A} is degenerate for $d = 1$, since its two eigenvectors coincide (Fig. 1) [22]. This implies, in particular, that every point flowing in the absorbing phase reaches F_{A} along the λ axis. It is not clear to us, at this point, what might be the physical signature of this for microscopic models.

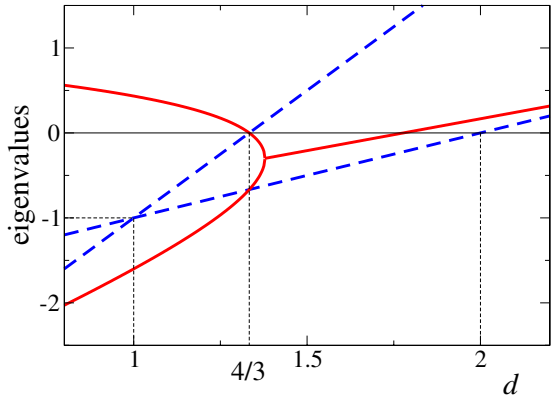

FIG. 1 (color online). Variation with d of the eigenvalues of the fixed points in the lowest-order LPA. Blue dashed lines: pure annihilation fixed point F_{A}. Red solid lines: nonperturbative fixed point F^* whose eigenvalues are both negative for $\frac{4}{3} < d < 1.3784\ldots$ and complex-conjugated at larger d (only the real part is plotted).

We now report on the results obtained for truncations of the potential u that go far beyond the simplest truncation described above. Of course, there exist many ways to organize a polynomial expansion of u around $\psi = 0$, $\bar{\psi}^2 = 1$ in terms of the three quadratic invariants $\psi\bar{\psi}$, $1 - \bar{\psi}^2$, ψ^2 which abides the Z_2 symmetry of the PC/GV class. Equivalently, because of Eq. (8), one can use any basis spanned by monomials in ψ^2, $\psi\bar{\psi}$, and we have tried several choices for it. In all cases, we stress that the qualitative picture unveiled at the minimal level is preserved at higher orders. It turns out that the fastest convergence of the

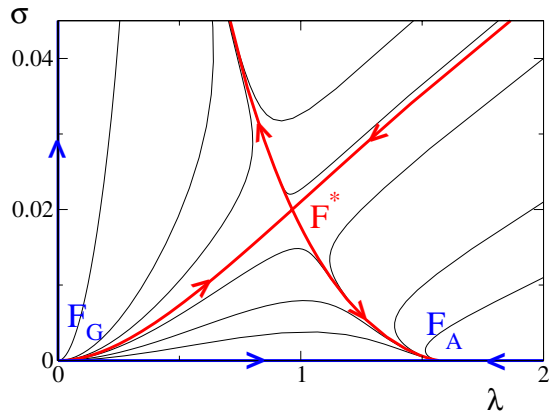

FIG. 2 (color online). Flow diagram of the lowest-order LPA in $d = 1$ (as usual, arrows represent the RG evolution as s is decreased towards the "infrared," macroscopic limit $k \to 0$).

TABLE I. Values of exponent ν with the order n of the LPA truncation (see text). The column "Min" refers to the minimal truncation (λ, σ). The last column is a conservative estimate taken from various Monte Carlo simulations.

n	Min	3	4	5	6	7	8	9	MC
ν	2.30	2.48	2.20	2.23	2.11	2.057	2.015	2.0017	1.85(10)

exponent ν is obtained with the basis, which is also the more transparent from the physical point of view. At order "n," we consider all possible branching $(n - 2m)A \to nA$ and annihilating reactions $nA \to (n - 2k)A$ involving at most n particles. These elementary reactions, respectively, correspond to all the terms $(1 - \bar{\psi}^{2m})(\psi\bar{\psi})^{n-2m}$ and $(1 - \bar{\psi}^{2k})\psi^n\bar{\psi}^{n-2k}$, and are anyhow ineluctably generated under renormalization. Table I shows the exponent ν computed up to order 9. The convergence is rather good, and we deduce that within the LPA $\nu = 2 \pm 0.1$, already in fair agreement with the results obtained in numerical simulations. We finally report a rather surprising fact: at all orders of the truncation, the dimension at which F^* and F_A cross is $d = \frac{4}{3}$. On the other hand we know, from the perturbation expansion performed around $d = 2$, that this dimension is no longer $\frac{4}{3}$ at two-loop order so that there is no reason to believe that this value is an exact result. Thus the deviation from $\frac{4}{3}$ can come only from orders beyond the LPA in the derivative expansion.

To summarize, we have shown, within the local potential approximation of the NPRG, that the critical point of the PC/GV class of absorbing phase transitions is a genuinely nonperturbative fixed point, out of reach of perturbative methods. This result will have to be refined within higher-order approximation of the derivative expansion, so that a full set of critical indices can be estimated, not just the correlation length exponent ν. Finally, ongoing work aims at a full numerical simulation of Eq. (7) (for the current problem and similar ones, such as the directed percolation class), which should allow to fully confirm these results and also offer access to other currently much debated reaction-diffusion problems, such as the pair contact process with diffusion [3].

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[1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, New York, 2002), 4th ed.
[2] H. Hinrichsen, Adv. Phys. **49**, 815 (2000); G. Ódor, Rev. Mod. Phys. **76**, 663 (2004).
[3] M. Henkel and H. Hinrichsen, J. Phys. A **37**, R117 (2004).
[4] L. Canet, H. Chaté, and B. Delamotte, Phys. Rev. Lett. **92**, 255703 (2004).
[5] J. L. Cardy and U. C. Täuber, J. Stat. Phys. **90**, 1 (1998).
[6] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A **17**, L105 (1984).
[7] I. Jensen, J. Phys. A **26**, 3921 (1993); D. Zhong and D. ben Avraham, Phys. Lett. A **209**, 333 (1995).
[8] J. Kockelkoren and H. Chaté, Phys. Rev. Lett. **90**, 125701 (2003).
[9] M. H. Kim and H. Park, Phys. Rev. Lett. **73**, 2579 (1994); N. Menyhárd, J. Phys. A **27**, 6139 (1994); W. Hwang *et al.*, Phys. Rev. E **57**, 6438 (1998); N. Inui and A. Y. Tretyakov, Phys. Rev. Lett. **80**, 5148 (1998).
[10] Earlier arguments supporting this viewpoint can also be found in H. Hinrichsen, Phys. Rev. E **55**, 219 (1997); W. Hwang and H. Park, Phys. Rev. E **59**, 4683 (1999); N. Menyhárd and G. Ódor, Braz. J. Phys. **30**, 113 (2000).
[11] O. Al Hammal *et al.*, Phys. Rev. Lett. **94**, 230601 (2005).
[12] I. Dornic *et al.*, Phys. Rev. Lett. **87**, 045701 (2001).
[13] M. Doi, J. Phys. A **9**, 1479 (1976); P. Grassberger and M. Scheunert, Fortschr. Phys. **28**, 547 (1980); L. Peliti, J. Phys. (Paris) **46**, 1469 (1985).
[14] In $d \geq 2$, the transition can be of two types, depending on parameters: either a direct, first-order-like GV transition, or a split to an Ising followed by a directed percolation transition. See [11] and also M. Droz, A. L. Ferreira, and A. Lipowski, Phys. Rev. E **67**, 056108 (2003).
[15] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[16] C. Bagnuls and C. Bervillier, Phys. Rep. **348**, 91 (2001).
[17] D. Litim, Phys. Rev. D **64**, 105007 (2001).
[18] L. Canet *et al.*, Phys. Rev. Lett. **92**, 195703 (2004).
[19] Following the line of reasoning of M. A. Muñoz, G. Grinstein, and Y. Tu, Phys. Rev. E **56**, 5101 (1997), it can be argued that the anomalous dimension of, e.g., the density field (in the PC language) is $\delta \approx 0.286\ldots$, δ being the exponent for homogeneous critical quenches. Although this is *a priori* a sizable correction, preliminary results for truncations beyond the LPA indicate that none of the qualitative features presented here are altered.
[20] Note that discarding the (somewhat trivial and arbitrary) dimensional part of Eq. (7), the flow equation of the effective potential is invariant under the exchange $\psi \leftrightarrow \bar{\psi}$. The equivalence between the PC and GV classes is thus preserved under renormalization.
[21] Equation (10) show a pole at $\sigma = \frac{1}{2}$. However, if we integrate the RG flow from a region of small σ, it never hits the pole. We also find that the relevant fixed point always lies in the small-σ region. In fact, the pole would be problematic only if we wanted to follow the flow of a microscopic action with $\sigma \geq \frac{1}{2}$ from $k = \Lambda$. In this case, our equations would have to be modified in the large k regime since then a cutoff function R_k diverging at $k = \Lambda$ would have to be taken, which would automatically eliminate the pole. Note that at any rate, such modifications do not change the fixed points, exponents, etc.
[22] Cardy and Täuber found from their RG analysis [5] of the "massive theory" performed at fixed dimension a flow in $d = 1$ that differs qualitatively from ours; albeit the two eigendirections around F_A were also parallel there.