

## Loss-Tolerant Optical Qubits

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We present a linear optics quantum computation scheme that employs a new encoding approach that incrementally adds qubits and is tolerant to photon loss errors. The scheme employs a circuit model but uses techniques from cluster-state computation and achieves comparable resource usage. To illustrate our techniques we describe a quantum memory which is fault tolerant to photon loss.

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Quantum logic gates can be built using linear optical networks in a scalable manner, as shown by Knill, Laflamme, and Milburn (KLM) [1]. A number of experimental efforts are currently focused on testing the building blocks of linear optical quantum computing (LOQC) [2–5]. However, optimism for large scale quantum computation based on LOQC has been tempered by the major overheads inherent in the KLM scheme and the high detector and source efficiencies apparently required [1]. For example, KLM estimate that of the order of 10 000 operations are needed per basic gate, and an efficiency threshold of  $\geq 99\%$ .

An alternative approach to implementing LOQC was proposed by Nielsen [6] and further developed by Browne and Rudolph [7] (see also [8] for related work). This approach combines the model of cluster-state quantum computation [9] with the nondeterministic gates presented by KLM, and achieves a very significant reduction in the overheads. The fault tolerance of the scheme has also been studied [10].

In this Letter we present a new approach to LOQC based on an incremental parity encoding [11]. Our method combines ideas from both the KLM and the cluster-state approaches. Parity encoding was used in the original KLM proposal to protect against both teleporter failures (i.e., the nondeterminism of the gates) and photon loss. By using parity encoding but reencoding incrementally (instead of by concatenation) we can obtain the reduction in overheads characteristic of the cluster-state approach while retaining the photon loss tolerance of KLM.

In particular we will describe a quantum memory which is fault tolerant [12] to photon loss based on single photon sources and detectors, linear optics, and feedforward. Though our techniques for detecting and correcting loss are themselves also subject to loss, above a particular threshold efficiency the effect of loss can be negated to arbitrary accuracy. A previous description of an optical quantum memory based on error correction did not consider fault tolerance [13]. Although we specifically only consider memory, our construction is compatible with gate operations and thus can form a template for fault tolerant quantum computation with respect to photon loss. We will

deal with qubits in three different tiers of encoding: *physical* encoding, *parity* encoding, and *redundant* encoding. Parity encoding allows near deterministic operations and converts photon loss to bit-flip errors. The redundant encoding then allows recovery from these errors.

*Physical encoding.*—At the first tier are the basic physical states that we will use to construct qubits, these will be the polarization states of a photon so that  $|0\rangle \equiv |H\rangle$  and  $|1\rangle \equiv |V\rangle$ . The advantage of this choice in optics is that we can perform any single physical-qubit unitary *deterministically*, with passive linear optical elements. Of course, gates between *different* physical qubits in LOQC are nondeterministic.

*Parity encoding.*—At the second tier of encoding are *parity qubits* encoded across many *physical qubits*. We shall use the notation  $|\psi\rangle^{(n)}$  to mean the logical state  $|\psi\rangle$  of a qubit, which is parity encoded across  $n$  physical qubits, i.e.,  $n$  distinct spatial modes each containing 1 photon [14].

Specifically, the parity encoding is given by

$$\begin{aligned} |0\rangle^{(n)} &\equiv (|+\rangle^{\otimes n} + |-\rangle^{\otimes n})/\sqrt{2} \\ |1\rangle^{(n)} &\equiv (|+\rangle^{\otimes n} - |-\rangle^{\otimes n})/\sqrt{2}, \end{aligned} \quad (1)$$

where  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ . The main feature of this encoding is that a computational basis measurement of any one of the physical qubits will not destroy the logical state, but instead will reduce the level of encoding by one. A bit-flip correction may be needed as a function of the measurement result.

Two operations are easily performed on parity encoded states, one is a rotation by an arbitrary amount around the  $x$  axis of the Bloch sphere [i.e.,  $X_\theta = \cos(\theta/2)I + i \sin(\theta/2)X$ ] [15], which can be performed by applying that operation to any of the physical qubits; and the other is a  $Z$  operation, which can be performed by applying  $Z$  to *all* the physical qubits (since the odd-parity states will acquire an overall phase flip). A key operation we will use is the partial Bell state measurement [16,17]. This consists of mixing two physical qubits on a polarizing beam splitter followed by measurement in the diagonal-antidiagonal basis. A successful event occurs when a photon is counted at each output of the beam splitter. An

unsuccessful event occurs when both photons appear at one of the outputs. When successful, it projects onto the Bell states  $|00\rangle \pm |11\rangle$ . When unsuccessful, it projects onto the separable states  $|01\rangle$  and  $|10\rangle$ , thus measuring the qubits in the computational basis. This operation can be used to add  $n$  physical qubits to a parity encoded state using a resource of  $|0\rangle^{(n+2)}$ . We will refer to this as type-II fusion ( $f_{II}$ ) [7]. The result of type-II fusion is

$$f_{II}|\psi\rangle^{(m)}|0\rangle^{(n+2)} \rightarrow \begin{cases} |\psi\rangle^{(m+n)} & (\text{success}) \\ |\psi\rangle^{(m-1)}|0\rangle^{(n+1)} & (\text{failure}) \end{cases}. \quad (2)$$

When successful (probability  $1/2$ ), the length of the parity qubit is extended by  $n$ . A phase flip correction may be necessary depending on the outcome of the Bell measurement. If unsuccessful a physical qubit is removed from the parity encoded state, and the resource state is left in the state  $|0\rangle^{(n+1)}$  (which may be recycled). This encoding procedure is equivalent to a gambling game where we randomly either lose one level of encoding, or gain  $n$ . Clearly if  $n \geq 2$  this is a winning game.

The remaining gates in order to achieve a universal gate set (a  $Z_{90}$  and a CNOT gate), can be efficiently performed on the parity encoded states by making use of the encoder above and will be described elsewhere [18]. The resource overhead for performing gates in this way is approximately equal to the best quoted for cluster-state encoding [7].

*Redundant encoding.*—The parity encoding has two purposes. First, the nondeterministic gates which we will employ, fail by measuring the qubit in the computational basis (hence this code enables recovery from gate failures). Second, loss of a photon can be considered a computational basis measurement in which we did not learn the answer. Thus upon loss of a photon we know that the remaining state is at worst a bit flipped version of the original. The final level of encoding is a redundancy code which enables recovery from this possibility of a bit flip:

$$|\psi\rangle_L = \alpha|0\rangle_1^{(n)}|0\rangle_2^{(n)} \dots |0\rangle_q^{(n)} + \beta|1\rangle_1^{(n)}|1\rangle_2^{(n)} \dots |1\rangle_q^{(n)}. \quad (3)$$

We can create an “encoder” gate that correctly encodes a parity qubit by simply fusing a more complicated resource state onto the parity qubit, namely,  $|0\rangle|0\rangle_1^{(n)}|0\rangle_2^{(n)} \dots |0\rangle_q^{(n)} + |1\rangle|1\rangle_1^{(n)}|1\rangle_2^{(n)} \dots |1\rangle_q^{(n)}$ . We attempt type-II fusion of this resource onto the parity qubit,  $|\psi\rangle^{(n)}$ , repeating until successful (on average twice) giving the (phase-flip corrected) result

$$\alpha[|0\rangle^{(n-k)}|0\rangle_1^{(n)} \dots |0\rangle_q^{(n)} + |1\rangle^{(n-k)}|1\rangle_1^{(n)} \dots |1\rangle_q^{(n)}] + \beta[|1\rangle^{(n-k)}|0\rangle_1^{(n)} \dots |0\rangle_q^{(n)} + |0\rangle^{(n-k)}|1\rangle_1^{(n)} \dots |1\rangle_q^{(n)}], \quad (4)$$

where  $0 < k < n - 1$  is the number of unsuccessful attempts made before fusion was achieved. This state is made up of  $qn$  “new” photons introduced by the resource and  $n - k$  of the “old” photons that made up the parity

qubit. By measuring the old photons in the computational basis and making a bit flip (on all new parity qubits if needed), we obtain the expected encoded state [Eq. (3)].

*Loss-tolerant qubit memory.*—A schematic of the memory circuit for the example of a 2 qubit redundancy code is shown in Fig. 1. The basic idea is as follows. The logical qubit is held in memory for some time, during which a photon may be lost. The logical qubit is then taken out of memory and one of its constituent parity qubits,  $P2$ , is sent into the encoder described above. The encoder performs two tasks: (i) it adds another level of redundancy encoding to the logical qubit and; (ii) it determines if a photon has been lost, without determining the logical value of the qubit. Figure 1(a) shows the procedure if no photons are found to have been lost. The state straight after the encoder is  $|\psi\rangle_L = \alpha|0\rangle_1^{(n)}|0\rangle_2^{(n)}|0\rangle_3^{(n)} + \beta|1\rangle_1^{(n)}|1\rangle_2^{(n)}|1\rangle_3^{(n)}$ . The other parity qubit,  $P1$ , is now measured in the diagonal basis  $|0\rangle^{(n)} \pm |1\rangle^{(n)}$ . This disentangles it from the other parity qubits which are returned to the state of Eq. (3) by the possible application of a phase flip (dependent on the outcome of the measurement on  $P1$ ). They are then returned to memory as shown.

Figure 1(b) shows the procedure when the encoder finds a photon missing in  $P2$ . Now the encoded state may have suffered a bit flip and we may have the state  $|\psi\rangle_L = \alpha|0\rangle_1^{(n)}|1\rangle_2^{(n)}|1\rangle_3^{(n)} + \beta|1\rangle_1^{(n)}|0\rangle_2^{(n)}|0\rangle_3^{(n)}$ . However, recovery is possible by now measuring the modes produced by the encoder in the diagonal basis. This disentangles  $P1$  from the other parity qubits without disturbing its logical value. Importantly, the correct  $P1$  is obtained regardless of whether a bit flip occurred to  $P2$ , though again a phase flip on  $P1$  may be required dependent on the outcome of the diagonal basis measurements. Finally,  $P1$  is put through an encoder, sent back to memory and the sequence is repeated. This will correct photon loss errors in which up to a single photon is lost per sequence. Higher levels of loss can be tolerated by increasing the size of the redundancy code placed in memory and generalizing the protocol. For example, a 3 qubit code could be kept in memory and 3 qubit encoders used. Then two loss events could be tolerated with recovery from the third qubit. We will describe how the various operations required for the memory circuit can be achieved using only linear optics, feed-forward, and Bell state resources.

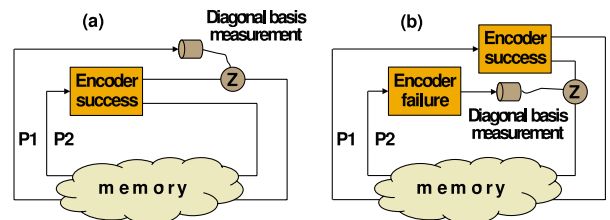


FIG. 1 (color online). A schematic of memory circuit.

*Threshold.*—First, consider the effect of photon loss in the encoder. If a loss event occurs in the fusion process, that is, only one photon is detected when a fusion is attempted, then the process is aborted. The presence of the redundancy code allows the following recovery. One of the remaining old photons is measured in the diagonal basis. This disentangles the entire parity qubit on which the encoder was attempted from the other parity qubit as described earlier. If fusion is successful, but a loss occurs while measuring the old photons in the computational basis, then measurement of any one of the remaining old physical qubits (or indeed one from each of the new pair of encoded parity qubits) will disentangle the other parity qubit which can then be reencoded.

The probability that a parity qubit will be successfully encoded, without photon loss, is given by  $P_{Q_s} = \sum_{i=1}^{n-1} (\frac{1}{2} \eta_1 \eta_2)^i \eta_1^{n-i}$ , where the size of the original parity qubit is  $n$  and the probability of detecting an old photon is given by  $\eta_1 = \eta_d \eta_s \eta_m$ , for a detector efficiency of  $\eta_d$ , a photon source efficiency of  $\eta_s$ , and a memory efficiency of  $\eta_m$ . The probability of successfully detecting a new photon is given by  $\eta_2 = \eta_d \eta_s$ . The photon source efficiency appears in the detection efficiency of an old photon because a photon may have been missing from the resource state used in the previous encoding sequence. Note that the fusion process will succeed or fail with probability  $\eta_1 \eta_2 / 2$  and detect a photon loss with probability  $1 - \eta_1 \eta_2$ .

Now let us consider the case of complete (unrecoverable) failure. This will occur if there is a sequence of fusion failures and photon loss events which result in all of the parity qubit component photons being lost without a successful disentangling operation being carried out. The probability of this occurring is given by

$$P_{ff} = \sum_{j=1}^{n-1} \left(\frac{1}{2} \eta_1 \eta_2\right)^{j-1} (1 - \eta_1 \eta_2) (1 - \eta_1)^{n-j} + R \sum_{j=0}^{n-2} \left(\frac{1}{2} \eta_1 \eta_2\right)^{j+1} \sum_{k=0}^{n-2-j} \eta_1^k (1 - \eta_1)^{n-1-j-k} + \left(\frac{1}{2} \eta_1 \eta_2\right)^{n-1} (1 - \eta_1). \quad (5)$$

Where  $R = \sum_{k=1}^q qk (1 - \eta_2)^{kn} [1 - (1 - \eta_2)^n]^{q-k}$  and takes into account failure to decouple using the new parity qubits also (measuring the components in diagonal basis). That leaves the probability that a photon loss occurs in the encoding of one parity qubit, but that we successfully disentangle it from the other parity qubit in the redundancy code:  $P_{Q_f} = 1 - P_{Q_s} - P_{ff}$ .

We can now calculate the threshold for the memory circuit. There are two ways in which the circuit can succeed. First, one of the parity qubits can be encoded without photon loss and then successfully disentangled from the other. This will occur with probability  $P_{Q_s} [1 - (1 - \eta_1)^n]$ . Second, a parity qubit can suffer photon loss but be suc-

cessfully disentangled, whereupon another parity qubit is successfully reencoded. This will occur with probability  $P_{Q_f} P_{Q_s}$ . Thus the probability of one successful sequence of the memory circuit for  $q$  parity qubits is

$$P_E = \sum_{j=0}^{q-1} P_{Q_f}^j P_{Q_s} [1 - (1 - \eta_1)^n]^{q-1-j}. \quad (6)$$

Although for fixed  $n$ ,  $\lim_{q \rightarrow \infty} P_E = 0$  and for fixed  $q$ ,  $\lim_{n \rightarrow \infty} P_E = 0$ , numerical investigations indicate that it is still possible to find  $n$  and  $q$  so that  $P_E$  approaches one.

The optimal  $q$  can be found from  $\frac{d}{dq} P_E = 0$  and using this value numerically we find that  $P_E$  approaches one for increasing  $n$  provided the threshold  $\eta > 0.82$  is satisfied. For efficiencies above about 0.96 a polynomial overhead in the code size results in an exponential decrease in the failure probability  $(1 - P_E)$ . For lower efficiencies the overhead is exponential. In Fig. 2 we show the behavior of  $P_E$  for optimal  $q$  as a function of  $\eta$  and  $n$ .

*Resources.*—We now discuss the creation of the resource states used to implement our memory circuit and hence the overheads needed. To this end we introduce a second operation, the *single-rail* partial Bell measurement [1]. This is achieved by mixing one of the polarization modes from each of 2 physical qubits on a beam splitter and counting photons at the outputs. A successful event occurs when one and only one photon is counted, otherwise it is unsuccessful. When successful it projects onto single-rail Bell states in which a logical zero is represented by the vacuum and a logical one by a single photon state. In terms of dual rail qubits its effect is to project onto the states  $(|H\rangle\langle HH| + |V\rangle\langle VV|)/\sqrt{2}$  or  $(|H\rangle\langle HH| - |V\rangle\langle VV|)/\sqrt{2}$  when successful, and measures each qubit in the computational basis when it fails. We will refer to this operation as type-I fusion, ( $f_I$ ) [7].

We will take as our basic resource the Bell state  $|0\rangle^{(2)}$ . Nondeterministic sources for such states are currently available and considerable effort is being made to create deterministic, or at least heralded sources of these states. To create the state  $|0\rangle^{(3)}$ , two  $|0\rangle^{(2)}$  can be fused together using the  $f_I$  gate. When successful, the  $|0\rangle^{(3)}$  state is

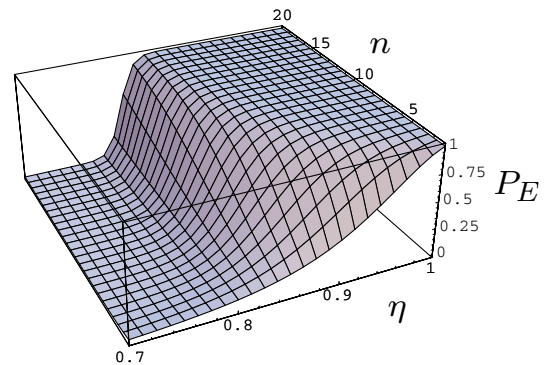


FIG. 2 (color online).  $P_E$  for optimal  $q$ .

produced, when unsuccessful, both Bell states are destroyed. With  $f_I$  functions with a probability of 1/2, on average two attempts are necessary, so, on average each  $|0\rangle^{(3)}$  consumes  $4|0\rangle^{(2)}$ .

Once there is a supply of  $|0\rangle^{(3)}$  states, either  $f_I$  or  $f_{II}$  can be used to further build up the resource state via

$$(H \otimes H)f_I H |0\rangle^{(n)} |0\rangle^{(m)} \rightarrow \begin{cases} |0\rangle^{(m+n-1)} & \text{(success)} \\ - & \text{(failure)} \end{cases}. \quad (7)$$

and Eq. (2). Using  $f_I$  with Hadamard gates has the advantage of losing only a single qubit from the input states, but the disadvantage of completely destroying the encoding in both input states in the event of failure. Using  $f_{II}$  to join the input states is at the expense of losing two of the initial qubits. There are two advantages to using  $f_{II}$ —first, in the case of failure, we do not destroy the encoding so far produced, just reduce this encoding by one and; second, the operation is “fail safe” in that a detection loss event is immediately recognizable as a failure (as 2 photons will not be counted) in contrast to  $f_I$  where photon loss can lead to a false positive.

We can avoid the problem of the  $f_I$  failure mode in the following way. If  $f_I$  gives a false positive it means that the mode exiting the fusion gate does not contain a photon. Thus our supply of  $|0\rangle^{(3)}$  states each have one “suspect” mode which may be vacuum. We now simply fuse two  $|0\rangle^{(3)}$  with  $f_I$  to form a  $|0\rangle^{(5)}$  using the suspect modes as the fusion point. We now are able to produce a supply of  $|0\rangle^{(5)}$  states which again have one suspect mode each. Finally, we use  $f_{II}$  to fuse two  $|0\rangle^{(5)}$  to produce a  $|0\rangle^{(8)}$ , once again using the suspect modes as the fusion point. This final fusion can not give a positive if a photon had been lost in either of the previous fusion events. In this way we can reliably produce the resource state  $|0\rangle^{(8)}$  regardless of detection efficiency. Of course missing photons due to finite source efficiency can still occur and are accounted for by  $\eta_s$ .

Using this approach and recycling  $f_{II}$  failures carries an average cost of approximately  $44|0\rangle^{(2)}$  per  $|0\rangle^{(8)}$  where we have assumed high detection and source efficiencies. Producing the encoder resource requires first the production of a  $|0\rangle^{(3)}$  onto which two  $|0\rangle^{(8)}$  are fused using  $f_{II}$ . A simple recycling strategy leads to a cost of approximately  $169|0\rangle^{(2)}$  (not necessarily optimal). Increasing the redundancy in the encoder resource requires only a linear overhead, i.e., the resource state for a  $q$ -fold redundancy encoder costs approximately  $(q-1) \times 169|0\rangle^{(2)}$ . Increasing  $n$  similarly carries a linear overhead.

*Conclusion.*—In this Letter we have introduced optical qubits with fault tolerance to loss under linear optical manipulations. We numerically determine the threshold for an optical memory based on these qubits to be 82% efficiency. That is, in principle for efficiencies higher than

this threshold, it is possible to find a suitable encoding such that the probability of a successful sequence of the quantum memory is arbitrarily close to 1. If we restrict ourselves to two parity qubits, each encoded across five physical qubits, and ask only when our quantum memory works with higher probability than a passive memory, then the answer is that the efficiencies of the sources and detectors must exceed 96%. In principle demonstrations of the parity code have been made [19,20]. Given the rapid improvements in source and detector technology, demonstrations of fault tolerance should be realistic in the medium term.

We have shown that a dramatic reduction in resource usage and increase in the threshold over the original scheme is possible with our parity encoding. Although we have only specifically discussed a quantum memory, the techniques can be generalized to include gate operations. We expect a number of the techniques described here could also be useful in optical quantum information processing with nonlinearities and other quantum information platforms.

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