Trading Interactions for Topology in Scale-Free Networks

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Scale-free networks with topology-dependent interactions are studied. It is shown that the universality classes of critical behavior, which conventionally depend only on topology, can also be explored by tuning the interactions. A mapping, $\gamma' = (\gamma - \mu)/(1 - \mu)$, describes how a shift of the standard exponent γ of the degree distribution $P(q)$ can absorb the effect of degree-dependent pair interactions $J_{ij} \propto (q_i q_j)^{-\mu}$. The replica technique, cavity method, and Monte Carlo simulation support the physical picture suggested by Landau theory for the critical exponents and by the Bethe-Peierls approximation for the critical temperature. The equivalence of topology and interaction holds for equilibrium and nonequilibrium systems, and is illustrated with interdisciplinary applications.

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In this Letter, we pose and answer the following fundamental questions. What are the relevant variables that determine the universality classes of critical behavior in networks? Is the conventional classification based on the exponent γ of the distribution function of network connections $P(q)$ sufficient? What is the effect on the critical behavior of interactions that depend on the connectivity? Do the interactions give rise to new relevant variables or can these be "transformed away" topologically? We focus on scaling arguments and analytic approaches [1].

Nowadays, scale-free networks enjoy a lot of attention due to their ubiquitous occurrence and the suitability of modern computational techniques for understanding their properties and predicting their behavior [2]. These networks typically have simple topology. The usual notions of spatial coordinates and dimensionality are not present; e.g., while the "volume" scales as the number of nodes N, the largest distance or "diameter" grows no faster than $\log N$, suggesting infinite "dimension." This is why mean-field or Landau theories are successful.

Understanding critical phenomena in scale-free networks is of fundamental and practical relevance. There is a surprising diversity of "mean-field" universality classes, which are topology dependent [3]. However, for most real networks and for the simplest interactions between nodes, the critical point is inaccessible (e.g., $T_c = \infty$). We propose a way around this by tuning the interactions.

Without loss of generality, consider first the context of opinion formation in "sociophysics" [4]. The network consists of people. Opinions are represented by spin orientations, and communication by pair interactions. The statistical physics of the formation of a common opinion is akin to that of spontaneous symmetry breaking below a critical temperature T_c in a Hamiltonian model for equilibrium cooperative phenomena [5].

Putting Ising (or Potts) spins s_i on the nodes $i = 1, \ldots, N$ and ferromagnetic interactions J on the edges leads to interesting critical behavior, different from its counterpart in lattice spin models [3,6]. For the standard scale-free distribution $P(q) \propto q^{-\gamma}$ of the number of connections or "degree" of a node, the decay exponent γ is the key parameter distinguishing the universality classes. Standard mean-field critical behavior ($\alpha = 0$ for the specific heat, $\beta = 1/2$ for the order parameter, and $\gamma_{\text{sus}} = 1$ for the susceptibility) is predicted for $\gamma > 5$. Logarithmic corrections appear at $\gamma = 5$. For $3 < \gamma < 5$ the critical exponents α and β are nonuniversal; they depend on γ. The susceptibility exponent $\gamma_{\text{sus}} = 1$ is superuniversal. Conventional finite-size scaling (as a function of network size N) predicts the familiar constraint $\alpha + 2\beta + \gamma_{\text{sus}} = 2$. For $\gamma \leq 3$, however, the framework of finite-temperature critical phenomena breaks down as T_c moves to infinity.

Which values of γ apply to real scale-free networks? Most studied ones, including WWW, Internet, collaboration, citation, cellular, ecological, or linguistic networks have $2 \leq \gamma \leq 3$. The Barabási-Albert (BA) network, grown by preferential attachment, has $\gamma = 3$.

The initial motivation for introducing degree-dependent interactions $J_{ij} \equiv J(q_i, q_j)$ was to prevent T_c from diverging in the Ising model on the BA network [5] by compensating high connectivity with weak interaction through $J_{ij} \propto 1/\sqrt{q_i q_j}$ [7]. Further work has led us to observe that this interaction is equivalent to a q-independent J, provided γ is shifted from 3 to 5. A Landau-theoretic argument and a mean-field ansatz generalize this equivalence (see further) and provide the insight that interactions that depend on connectivity effectively modify the topological distribution $P(q)$.

We introduce a family of interactions parametrized by an exponent μ,

$$J_{ij} = J\langle q\rangle^{2\mu}(q_i q_j)^{-\mu}, \tag{1}$$

where $\langle\cdot\rangle$ denotes the average over the degree distribution $P(q)$. Generalizing the Landau theory of Goltsev *et al.* [3], we obtain the following form, in zero external field, for the "free energy" Φ as a function of the order parameter $x \equiv \sum_i E_T(s_i)/N$, with $E_T(\cdot)$ the thermal average:

$$\Phi(x) = \langle \phi(q^{-\mu}x, q^{1-\mu}x)\rangle, \tag{2}$$

where the first argument of the constrained free energy ϕ is the rescaled order parameter, and the second one is the rescaled effective field acting on a node with q connections. Both rescalings, by a factor $q^{-\mu}$, come from the q_i and q_j dependence of J_{ij}. The crucial assumption [3] is that $\phi(y, z)$ can be written as a power series in y and z. The singularity structure of $\Phi(x)$, leading to fascinating deviations from standard mean-field behavior, is then induced by the fact that the moments $\langle q^n\rangle$ diverge for $n \geq n_c = \gamma - 1$. This in turn leads to a divergence of the coefficient f_n of x^n in the free energy. Thus, in this system nonclassical critical behavior is possible, not due to spatial fluctuations of the order parameter (since space is effectively infinite dimensional), but due to the scale-free character of the degree distribution.

Our main result is that a network with exponents (γ, μ) can be mapped onto one with $(\gamma', \mu' = 0)$, in the sense that both are in the same universality class of critical behavior. The latter has constant couplings, independent of q. We find the exponent relation

$$\gamma' = (\gamma - \mu)/(1 - \mu), \tag{3}$$

which can be proven as follows. The critical value $n = n_c$ for diverging coefficients f_n in the free energy must be the same in order for the two models to be equivalent. Since the leading moment $\langle q^n\rangle$ in f_n in the model with $\mu = 0$ gets replaced by $\langle q^{(1-\mu)n}\rangle$ in the model with $\mu \neq 0$, we obtain $n_c = \gamma' - 1 = (\gamma - 1)/(1 - \mu)$.

A more general "mean-field" proof, which does not rely on a free energy, runs as follows. Within a mean-field approach, the only way in which the degree q enters in physical properties is through the *quenched average interaction*, over all networks, between any two nodes with *fixed* degrees q_i and q_j. This average is $J_{ij}p_{ij}$, with $p_{ij} = q_i q_j/(\langle q\rangle N)$ the probability that i and j are connected. Note that even for constant J (case $\mu = 0$) the quenched average interaction is q dependent, while for $\mu = 1$ it is not. For $\mu \neq 0$, the q_i are transformed to $q_i' = q_i^{(1-\mu)}$, using (1). In order to retain the same physics, averages over the degree distribution must be invariant. This requires a distribution transformation,

$$P(q) = P'(q'(q))dq'(q)/dq, \tag{4}$$

from which (3) follows for scale-free (power-law) $P(q)$.

Consequently, for a scale-free network the range $\mu \in [2 - \gamma, 1]$ allows one to explore the whole range of universality classes uncovered in previous works. It is no longer necessary to vary the network topology. It suffices, for a fixed γ, to tune the form of the interaction. An important consequence is that for real scale-free networks, with typically $\gamma \leq 3$, one is no longer set back by an infinite critical temperature when putting on interactions. For the BA network the limit $\mu \to 1$ maps onto $\gamma' \to \infty$ and corresponds to the crossover to a thin-tailed degree distribution of, e.g., Poisson type. In the opposite limit, $\mu \to -1$, for which $\gamma' \to 2$, the threshold is reached beyond which the first moment $\langle q\rangle$ diverges in the equivalent network with constant couplings.

For locating the critical temperature, a Bethe-Peierls (BP) approximation, which emphasizes the local Cayley treelike structure of the network, normally gives a very reasonable first approximation [6]. The entropic contribution to the free energy is truncated to single-spin and pair terms [8]. Besides this standard assumption, we impose a new scaling relation on the local order parameter, $E_T(s_i)$, in a given network. This scaling is in harmony with the definition of the effective field acting on s_i in the Landau theory and is corroborated by Monte Carlo simulation (MCS). It reads

$$E_T(s_i) \approx q_i^{1-\mu}x/\langle q^{1-\mu}\rangle. \tag{5}$$

Using this to eliminate s_i in terms of q_i and x, one obtains the self-consistent BP equation near T_c,

$$\langle q\rangle \sum_q P(q)(q-1)q^{1-\mu} = \sum_{q_1} P(q_1)q_1^{2-\mu} \sum_{q_2} P(q_2)q_2\left(1 + \tanh\frac{J\langle q\rangle^{2\mu}}{k_B T q_1^{\mu} q_2^{\mu}}\right)^{-1}. \tag{6}$$

For the conventional case $\mu = 0$, we can extract from this the large-$\langle q\rangle$ approximation $k_B T_c/J \approx \langle q^2\rangle/\langle q\rangle - 1$, in agreement with exact results [6].

We now turn to illustrations of this framework and start with $\gamma = 3$ and $\mu = 1/2$ ("Special Attention Network" [7]). We discuss the critical point, specific heat, order parameter, and susceptibility.

The critical temperature versus average degree $\langle q\rangle$ is shown in Fig. 1, for $\langle q\rangle = 2$, 4, and 6. It has been derived by exact solution of the model using the replica technique. Since all couplings are "ferromagnetic," the replica symmetry is not broken. For comparison, the almost coincident results from the BP approximation (6) are also indicated. A good rule of

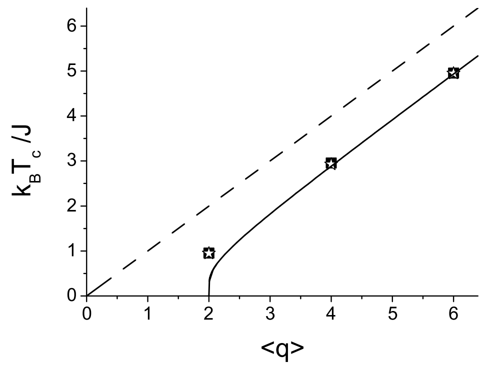

FIG. 1. T_c versus $\langle q\rangle$ for a scale-free network with $\gamma = 3$ and $\mu = 1/2$. BP estimates (solid squares) almost coincide with exact results (replica technique, open stars). The solid line is the lattice BP approximation for $Q = \langle q\rangle$, and the dashed line is the mean-field approximation.

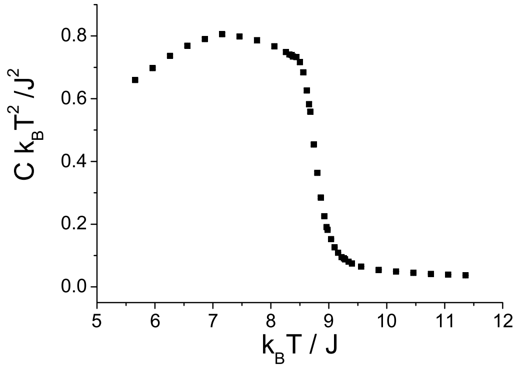

FIG. 2. Specific heat from MCS of BA networks ($\gamma = 3$), with $\mu = 1/2$ and $\langle q\rangle = 10$. The critical temperature in the large-N limit is $k_BT_c/J \approx 8.96$ [Eq. (6)]. For $N = 5600$, MCS of the susceptibility maximum indicates $k_BT_c(N)/J \approx 8.65$, well above the temperature of the specific heat maximum.

thumb is $k_BT_c/J \approx \langle q\rangle - 1$. Incidentally, this is also the result of the large-$\langle q\rangle$ expansion which can be derived from (6). Interestingly, the results are very close to the conventional BP approximation $J/k_BT_c = 0.5\ \ln[(Q-2)/Q]$ (thin solid line), for a regular lattice with coordination number Q, except for $Q = 2$. For completeness the mean-field conjecture [7] $k_BT_c/J = \langle q\rangle$ is also shown.

These calculations assume no correlations exist between the edges of the network. The rule of preferential attachment violates this assumption, so the BA network must be considered separately. For the special case $\langle q\rangle = 2$, a BA network differs strongly from a correlation-free network. The former consists of a single treelike structure *without* loops, whence $T_c = 0$. In contrast, an uncorrelated network with $\langle q\rangle = 2$ can consist of clusters, which can have loops, leading to a finite T_c, as replica technique and BP approximation predict.

Having located the critical temperature, and elucidated its nonuniversality through its dependence on $\langle q\rangle$ and on the network correlations, we now turn to more universal properties. The specific heat singularity for the Ising model on scale-free networks has a very interesting and subtle form [6]. For $\mu = 0$, it varies from a classic mean-field jump, predicted for $\gamma > 5$, to a continuous behavior but with diverging slope, for $4 < \gamma < 5$, and with continuous slope for $3 < \gamma < 4$. For $\mu = 1/2$, the equivalence relation [Eq. (3)] predicts, with $\tau = (T - T_c)/T_c$,

$$C_{\mathrm{sing}} \propto (\ln\tau^{-1})^{-1}. \tag{7}$$

The shape of this specific heat near T_c is astonishingly different from that of the conventional mean-field jump. Although the critical exponent α consistent with (7) is zero, as for standard mean field, the inverse logarithm ensures a *vanishing* jump, with diverging slope.

Evidence gathered from MCS and, more convincingly, the cavity method shows that (i) the specific heat $C(T)$ reaches a maximum well *below* T_c and (ii) the jump singularity that is apparent for small network size N closes slowly when N is increased. These findings are consistent with (3), and suggest that, for $\gamma = 3$ and $\mu = 1/2$, the network indeed maps onto one with $\gamma = 5$ and $\mu' = 0$.

Figures 2 and 3 illustrate these results for the specific heat, for a BA network with $\mu = 1/2$. Shown are MCS (Fig. 2) for $\langle q\rangle = 10$ and $N = 5600$, and the cavity method (Fig. 3) for $\langle q\rangle = 4$, applied to the sequence $N = 100, \ldots, 10^6$. The latter technique clearly suggests the slow closing of the specific heat jump, for $N \to \infty$.

We have also computed the order parameter (''magnetization'') and susceptibility singularities near T_c and find, for $\mu = 1/2$, that the results are consistent with $\beta = 1/2$ and $\gamma_{\mathrm{sus}} = 1$. Again, these values agree with the predictions from the mapping (3). It should be remarked that for $\gamma = 5$ and $\mu = 0$ a logarithmic correction factor is predicted for the order parameter, which, however, cannot

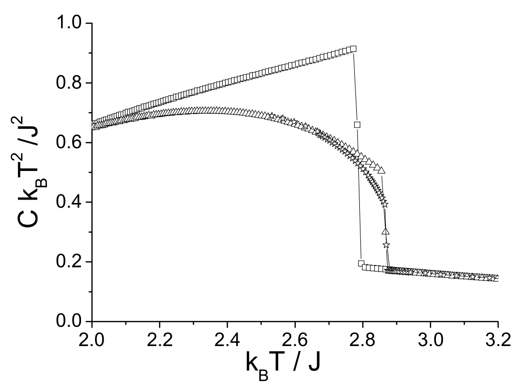

FIG. 3. Specific heat from the cavity method for BA networks, with $\mu = 1/2$ and $\langle q\rangle = 4$. Shown are data for $N = 10^2$ (squares), $N = 10^4$ (triangles), and $N = 10^6$ (stars). Lines are guides to the eye. The finite-size critical temperature (apparent from the jump in C) rapidly converges for large N, to $k_BT_c/J \approx 2.92$ [Eq. (6)]. Clearly, the maximum in C precedes T_c and the jump in C tends to close for large N.

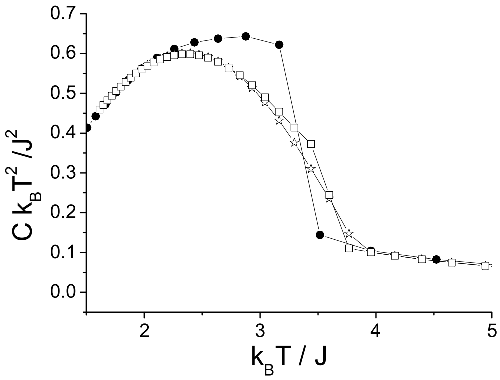

FIG. 4. Specific heat from the cavity method for BA networks, with $\mu = 1/3$ and $\langle q \rangle = 4$, for $N = 10^2$ (circles), $N = 10^3$ (squares), and $N = 10^5$ (stars). A linear T dependence develops for large N and meets the high-T background with a jump in slope at T_c. The finite-size critical temperature converges to $k_B T_c/J \approx 4.21$ [Eq. (6)].

easily be detected on top of the square-root singularity $\beta = 1/2$. Furthermore, our result for γ_{sus} is not discriminative, since $\gamma_{\mathrm{sus}} = 1$ is superuniversal, valid for all $\gamma > 3$ in the model with $\mu = 0$.

Further evidence for (3) is obtained for the interesting case $\gamma = 3$ and $\mu = 1/3$. A linear specific heat ($\alpha = -1$) and a linear order parameter singularity ($\beta = 1$) clearly emerge for large N in cavity method computations (Fig. 4). MCS of specific heat (Fig. 5), order parameter, and susceptibility supports this conclusion. The results agree with what is expected for $\gamma' = 4$ and $\mu' = 0$.

Additional evidence for (3) comes from a study of networks with $\gamma = 3$ and $\mu = 1$. Using MCS and cavity method, the standard mean-field jump of the specific heat is retrieved. This is consistent with a shift of γ to a value greater than 5 in the reference system with $\mu = 0$.

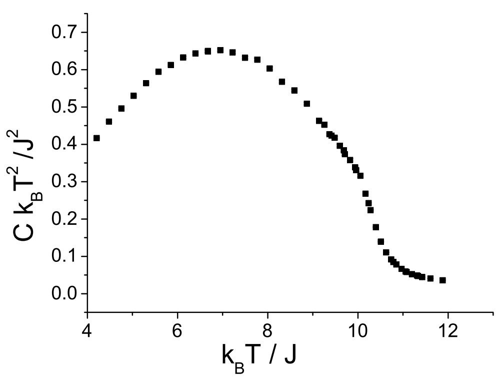

FIG. 5. Specific heat from MCS of BA networks with $N = 5600$, for $\mu = 1/3$ and $\langle q \rangle = 10$. The critical temperature in the large-N limit is at $k_B T_c/J \approx 11.33$ [Eq. (6)].

We now leave equilibrium statistical mechanics and focus on dynamical systems. In the contact process for disease spreading, each node can be either ill or healthy. An ill node can cure at rate 1, and a healthy node becomes infected at a rate which is λ times the number of ill neighbors. The model has a phase transition between an absorbing healthy state and an active state with a nonzero density of ill nodes, at some λ_c. On a scale-free network a finite λ_c is found for $\gamma > 3$. The critical exponents depend on γ in the range $3 < \gamma \leq 4$ and assume standard mean-field values for $\gamma > 4$ [9]. Generalizing the process to a degree-dependent infection rate, $\lambda_{ij} = \lambda \langle q \rangle^{2\mu} (q_i q_j)^{-\mu}$, we arrive again at (3). The exponent mapping appears to be quite generally valid [10].

In conclusion, static and dynamic order-disorder transitions on scale-free networks display singularities that depend on the network topology *and* on the form of the interactions. Connectivity-dependent interactions can be used as a probe of topology-dependent cooperative behavior. The exponent mapping (3) prescribes how to trade interactions for topology.

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