Cooling Mechanism for a Nanomechanical Resonator by Periodic Coupling to a Cooper Pair Box

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We propose and study an active cooling mechanism for the nanomechanical resonator (NAMR) based on periodical coupling to a Cooper pair box (CPB), which is implemented by a designed series of magnetic flux pluses threading through the CPB. When the initial phonon number of the NAMR is not too large, this cooling protocol is efficient in decreasing the phonon number by 2 to 3 orders of magnitude. Our proposal is theoretically universal in cooling various boson systems of a single mode. It can be specifically generalized to prepare the nonclassical state of the NAMR.

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Introduction.—Recently nanomechanical resonators (NAMRs) have been fabricated with high quality factors (from 10^2 to 10^5) and large fundamental frequencies (in the range of MHz–GHz) [1–3]. The NAMR has been shown as a good candidate for exploring various mesoscopic quantum phenomena at the boundary between classical and quantum realms. Until now NAMRs have been used in generating entangled states [4], demonstrating quantum nondemolition measurement [5] and progressive quantum decoherence [6], and implementing a two-qubit quantum gate [7].

The quantum nature of NAMRs has been exhibited by the accurate measurement near the standard quantum limit [8,9]. But in most cases, to fully utilize the quantum wealths provided by NAMRs it is necessary to cool the NAMR to its ground state. There have been some schemes for cooling NAMRs [10–12]. Some of them are based on coupling with Josephson junction (JJ) qubit and make use of feedback control and sideband cooling techniques. Characterized by the maximal ratio between the average number of phonons before and after cooling, the highest efficiency of some schemes [11,12] can be achieved when the initial number of phonons $N_{\rm th}$ of the NAMR is large enough ($N_{\rm th} \gtrsim 10^{-1}$). The cooling effect is evidently decreased as $N_{\rm th}$ gets very small.

Motivated by the existing investigations mentioned above, we suggest a straightforward mechanism to cool the NAMR. Our scheme is also based on the coupling with the Cooper pair box (CPB), which is considered as a controllable two-level system. Different from the existing schemes, our cooling protocol works efficiently in small $N_{\rm th}$ regime ($N_{\rm th} \lesssim 10^1$). In our scenario, the interaction takes place periodically between the CPB and the NAMR. Before the interaction takes place in each cycle, the CPB is always set to its ground state so that it can absorb some energy from the NAMR during the interaction period. A similar method has been used by to cool the microwave cavity [13,14]. In principle, the present proposal can be generalized for cooling any single mode boson system via the coupling with a two-level system.

One can intuitively compare our cooling mechanism with a classical analog. To cool a thermal box one can put a piece of ice into it and then drain the meltdown water. It will take away part of the heat in the box. Naively, one can freeze the drained water into ice outside the box in some way and then place it back into the box. Repeat this process again and again until the box reaches the desired temperature. In our scheme, the CPB prepared in the ground (excited) state can be imagined as the ice (water) and the NAMR as the thermal box in the classical analog. However, due to quantum coherence, the mechanism of our cooling protocol is not as naive as this "ice-and-box" analog, because the energy loss of the box is irreversible due to the second law of thermodynamics. The substantial difference of our protocol from the above classical analogue is the coherent oscillation of energy exchange between the qubit and the bosonic mode. In a sense, our scheme is more related to a type of quantum heat engine [15].

Model for our cooling protocol.—In our NAMR-CPB composite system shown schematically in Fig. 1, the NAMR is directly connected to a CPB consisting of two Josephson junctions. The external magnetic flux Φ_x threads through the SQUID, which can be used to adjust

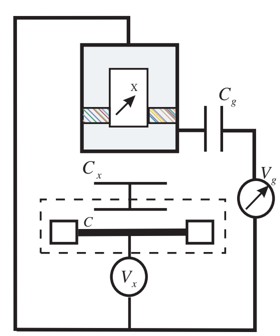

FIG. 1 (color online). The coupling system of the NAMR (within the rectangle of dashed lines) and the CPB: the bang-bang coupling is implemented by a predesigned periodic series of magnetic flux Φ_x threading through the CPB.

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its effective Josephson energy. C_J is the Josephson capacitance, V_g the tunable gate voltage, C_g the gate capacitance, V_x the bias voltage on the resonator, and C' the effective capacitance of the NAMR. C_x denotes the distribution capacitance between the CPB and the NAMR. Let $E_c = e^2/(2(C_x + C_g + C_J)$ be the charge energy unit and $n_g(x) = (C'(x)V_x + C_g V_g)/2e$ be the total gate charge.

The dependence of charging energy $4E_c(n_c - n_g(x))^2$ on x results in the coupling between the NAMR with free Hamiltonian $p^2/(2m) + m\omega_0^2 x^2/2$ and the CPB with controllable Josephson tunneling energy $-2E_J \times \cos(\pi\Phi_x/\Phi_0)\cos\theta$. Here, n_c denotes the number of the excess Cooper pair on the island while its conjugate variable is the phase difference θ of the two sides of each junction. Usually the fluctuation of x is much smaller than the distance d between the NAMR and the CPB. At the exact resonance point defined by $(C_x V_x + C_g V_g)/2e = N/2$ where N is odd, the CPB acts as a two-level system, a qubit. We denote two linearly independent charge states by $|1\rangle = |n_c = (N-1)/2\rangle$ and $|2\rangle = |n_c = (N-1)/2\rangle$. Under the two-level approximation and the rotation wave approximation (RWA), we write the above Hamiltonian as the Jaynes-Cummings (JC) form [12]

$$H = E_J \cos\left(\pi\frac{\Phi_x}{\Phi_0}\right)\sigma_z + \omega_0 a^\dagger a + g(a\sigma_+ + \text{H.c.}). \quad (1)$$

Here, $\sigma_z(\sigma_+, \sigma_-)$ are defined with respect to the new basis $\{|e\rangle = (|1\rangle + |2\rangle)/\sqrt{2}, |g\rangle = (|1\rangle - |2\rangle)/\sqrt{2}\}$; a $(a^\dagger)$ are the phononic creation and annihilation operators of the NAMR mode with the effective coupling constant $g = 4E_c n_x\sqrt{1/2m\omega_0}/d$ where $n_x = C_x V_x/(2e)$.

In our protocol, the above JC-type interaction is assumed to take place periodically with rate r_a. Then it is switched off after the duty cycle interval τ of the order of g^{-1}. This on-and-off switching can be realized by the magnetic flux Φ_x. In fact, at the exact resonance point, the JJ tunneling energy $E_J\cos(\pi\Phi_x/\Phi_0)$ is just the energy level spacing. It will be switched to the value resonant with the NAMR during the duty cycle τ and far-off resonant outside this period. The similar manipulation has been used to create the nonclassical photon state based on the superconductor devices [16].

To implement our proposal we have two main tasks to realize the periodical coupling: (a) Switching on and off the interaction between the CPB and the NAMR periodically, and (b) Preparing the CPB to its ground state before the interaction takes place in each cycle. Both tasks can be accomplished via tuning the magnetic flux Φ_x with time. In fact, the gate charge fluctuation induced relaxation rate $\Gamma(\omega) = \pi\alpha_g\omega[\coth(\omega/2k_BT) + 1]/2$ of the CPB at temperature T [12] can be well controlled by varying the flux Φ_x since $\omega = E_J\cos(\pi\Phi_x/\Phi_0)$. Here, k_B is the Boltzmann constant, α_g is about $2e^2R(C_x^2 + C_g^2)/(\pi(C_x + C_g + C_J)^2)$, and R the fluctuation impedance of V_g and V_x. Outside the duty cycle τ, one can switch the energy spacing ω of CPB to a large value to satisfy $\omega \pm \omega_0 \gg g$. In this case, the CPB-NAMR interaction is effectively switched off because of the far-off resonance (this can be deduced without RWA) while the decay process is enhanced. Thus the CPB can be prepared well in its ground state $|g\rangle$ for the upcoming interaction period.

Master equation approach in steady states.—We assume the coupling strength g is much stronger than $\Gamma(\omega_a)$ and κ where κ is the decay rate of NAMR, and the interaction period τ is so short that $\Gamma(\omega_a)\tau \ll 1$, $\kappa\tau \ll 1$. In this case both the decay of CPB during the duty cycle and the NAMR-environment coupling can be omitted. Therefore, if the interaction is switched on at instance t_l, the reduced density operator $\rho(t_l + \tau)$ of the NAMR after a time interval τ can be obtained through the action of the superoperator $M(\tau)$ on the reduced density operator $\rho(t_l)$ at instance t_l, i.e., $\rho(t_l + \tau) = M(\tau)[\rho(t_l)]$, which is defined as $M(\tau)[\rho(t_l)] = \text{Tr}_a[\exp(-i\hat{h}\tau)\rho(t_l) \otimes |g\rangle\langle g|\exp(i\hat{h}\tau)]$ Tr_a denotes tracing over the variables of CPB. $\hat{h} = ga\sigma_+ + \text{H.c.}$ is the JC-type Hamiltonian (1) at resonance in the interaction picture. Without any dissipation, the exact solution of the resonant JC model gives the explicit recursions

$$p_n(t_l + \tau) = |c_{g,n}(\tau)|^2 p_n(t_l) + |c_{e,n}(\tau)|^2 p_{n+1}(t_l) \quad (2)$$

for the diagonal elements $p_n = \langle n|\rho|n\rangle$ of $\rho(t_l + \tau)$ for the number $|n\rangle$ state of NAMR phonon. Here, $c_{e,n}(\tau) = \sin(g\tau\sqrt{n+1})$ and $c_{g,n}(\tau) = \cos(g\tau\sqrt{n})$ come from the exact solution of the resonant JC model.

With the presence of the dissipation of NAMR, the evolution of ρ can be depicted by the course gained master equation

$$\frac{d\rho}{dt} = r_a[M(\tau) - 1]\rho + L[\rho]. \quad (3)$$

The super operator L in the above equation is attributed to the dissipation and is defined as $L[\rho] = -(\kappa/2)N_{\text{th}}(aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger) - (\kappa/2)(N_{\text{th}} + 1) \times (a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a)$ where $N_{\text{th}} = [\exp(\omega_0/k_BT) - 1]^{-1}$ is the average number of phonons of NAMR at temperature T before cooling. The master equation (3) was initially presented for the case that the interaction between the two-level system and the single mode oscillator is "turned on" randomly [17,18]. In the case where the interaction in our scheme is periodically turned on, this equation can also lead to a correct stable solution.

Without detailed computations, the average number of phonons $\langle n\rangle_s = \text{Tr}[\rho_s a^\dagger a]$ of NAMR in the steady state ρ_s can be obtained from the above master equation:

$$\langle n\rangle_s = N_{\text{th}} - (r_a/\kappa)\Delta n, \quad (4)$$

where $\Delta n = \text{Tr}\{a^\dagger a[1 - M(\tau)]\rho_s\}$. Since $\text{Tr}\{a^\dagger a[1 - M(\tau)]\rho_s\} > 0$ can be proved with the definition of $M(\tau)$, we can conclude that $\langle n\rangle_s < N_{\rm th}$; i.e., the NAMR can always be cooled when the steady state is reached.

Dynamic process of cooling and the fluctuation of number of phonons.—The steady state solution of Eq. (3) gives the phonon population

$$p_n^s = p_0^s \prod_{l=1}^{n} \frac{N_{\rm th} l}{(N_{\rm th}+1)l + |c_{e,l-1}|^2 r_a/\kappa}. \qquad (5)$$

Here, p_n^s is nth diagonal element of ρ_s and p_0^s is determined by the normalization condition $\Sigma_{i=0}^{\infty} p_i^s = 1$ [17]. By virtue of numerical computations, we can describe exactly the evolution of vacuum state probability p_0 and average number $\langle n\rangle$ of phonons in detail. Figure 2 demonstrates these results with experimentally rational parameters $N_{\rm th} \approx 1.7$, $r_a/\kappa = 133$, and $g\tau = \pi/8$. It is shown that, at the time t_a satisfying $r_a t_a \sim 60$, the steady solution (5) is reached. In this sense the distribution of the number of phonons in the data does not vary significantly after time t_a.

Apparently, we can improve the cooling effect by increasing r_a. For a given value of r_a/κ, our cooling scheme works well when the average number of phonons $N_{\rm th}$ before cooling is small enough, i.e., $N_{\rm th} \ll r_a/\kappa$. Under this condition, we can achieve the maximum cooling effect by setting the duty cycle $\tau = \pi/(2g)$ so that $|c_{e,0}| = 1$. In this case the average number of phonons after cooling is

$$\langle n\rangle_s \approx p_1^s \approx N_{\rm th}\kappa/r_a. \qquad (6)$$

This result implies that the number of phonons is reduced by a factor of (r_a/κ). However, if $N_{\rm th}$ is comparable with (or larger than) $|c_{e,0}|^2 r_a/\kappa$, our scheme does not work well. For $N_{\rm th}+1 \gg r_a/\kappa$, we have $p_n \approx p_0[N_{\rm th}/(N_{\rm th}+1)]^n$, which implies that the average number of phonons $\langle n\rangle_s$ in the steady state is very close to $N_{\rm th}$, the number of phonons before cooling. The average number $\langle n\rangle_s$ of phonons after cooling is drawn against $N_{\rm th}$ in Fig. 3. It shows that, with $r_a/\kappa = 10^2$ and $|c_{e,0}| = 1$, we have $\langle n\rangle_s \approx 10^{-2}N_{\rm th}$ for $N_{\rm th} \lesssim 1$; with $r_a/\kappa = 10^3$ and $|c_{e,0}| = 1$, we have $\langle n\rangle_s \approx 10^{-3}N_{\rm th}$ for $N_{\rm th} \lesssim 10$ and $\langle n\rangle_s \approx 10^{-2}N_{\rm th}$ for $N_{\rm th} \lesssim 10^2$.

It is also noted that, since the interaction between the CPB and the NAMR is switched on and off again and again, and the master equation (3) is obtained via a coarse granulation approach, the number of phonons will have a fluctuation $\Delta n = {\rm Tr_N}\{a^\dagger a[1-M(\tau)]\rho_s\}$ even in the steady state. In the inset of Fig. 2, the fluctuation Δn is shown in the evolution curve of the number of phonons in the steady state.

Experimentally feasible predications.—We consider a NAMR with frequency $\omega_0 = 2\pi \times 10^2$ MHz (0.5 μeV) and quality factor $Q = 2\times 10^5$. Our cooling protocol can be realized with the following experimentally accessible parameters, the decay rate $\kappa = \pi \times 10^{-3}$ MHz, the Josephson energy $E_J \approx 4\pi \times 10^4$ MHz (100 μeV), the Coulomb charging energy $E_C \approx 320$ μeV (i.e., $C_\Sigma \approx 250$ aF, the mutual capacitance $C_x = C_g \approx 20$ aF), the impedances $R \approx 50\ \Omega$ and the gate voltage $V_x \approx 0.25$ V. With these parameters, the interaction strength g is estimated to be $2\pi \times 10$ MHz, the number of Cooper pairs n_x is about 15 and $\alpha_g \approx 1\times 10^{-4}$.

When the magnetic flux Φ_x in the Hamiltonian (1) is tuned to about $0.498\Phi_0$ on resonance, the interaction takes place. We set the interaction duration $\tau = 2.5\times 10^{-8}$ s so that $|c_{e,0}(\tau)| = 1$. If the temperature T is 0.01 K, during the duty cycle, the decay rate $\Gamma(\omega_0)$ of CPB is about 0.56 MHz. Just after the interaction, the magnetic flux should be turned off to maximize the energy spacing. Then the CPB will decay to its ground state by a rather large decay rate $\Gamma(E_J) \approx 40$ MHz. The CPB then decays quickly in $\tau' = 0.275$ μs. This elementary procedure is repeated at a frequency $r_a \approx 3$ MHz. Namely, the interaction takes place every $T_1 = 1/r_a = 0.3$ μs. The time is sufficiently long for completing the interaction and the relaxation in each elementary procedure. Therefore, omitting the dissi-

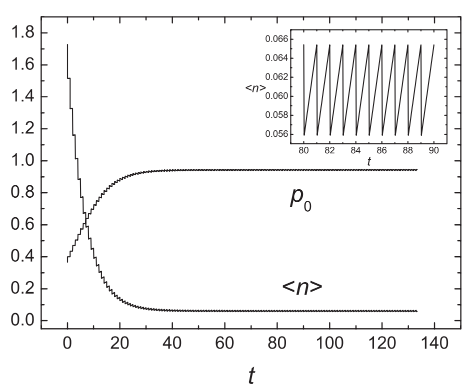

FIG. 2. The time evolution of the average phonon number $\langle n\rangle$ and the vacuum state probability p_0. The unit of time is $1/r_a$. The fluctuation of the phonon number in the steady state is shown in the inset.

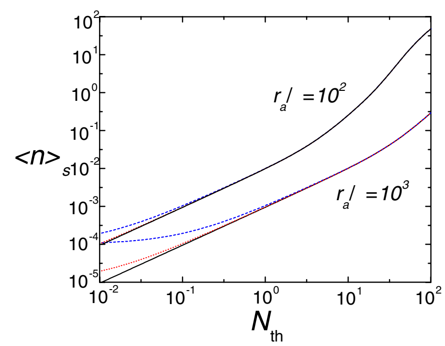

FIG. 3 (color online). The cooling effect diagram. Here we assume $\alpha_g \approx 1\times 10^{-4}$. The finial phonon number $\langle n\rangle_s$ is drawn as a function of the initial phonon number $N_{\rm th}$ when $r_a/\kappa = 10^2$ or 10^3. In the solid (black) line, the thermal excitation of the CPB is not considered. In the dashed (blue) line, we assume the thermal excitation probability of the CPB is $p = 10^{-4}$. In the dotted (red) line, we assume $p = 10^{-5}$.

pation effect of the CPB, we can estimate from Eq. (6) that the number of phonons after cooling has the magnitude of $\langle n \rangle_s \sim N_{\rm th}/(r_a/\kappa) \sim 10^{-3} N_{\rm th}$.

In the above discussion, the parameter of crucial importance κ is determined by the quality factor Q and the resonance frequency ω_0 of the NAMR. As illustrated above, for $Q \sim 2 \times 10^4$, we end up with $r_a/\kappa = 10^2$, and the cooling effect would be much worse. A careful analysis of the cooling cycle reveals that the cooling effect is also influenced by the dissipation of the CPB during the duty cycle. For high initial temperatures, we can replace the factor $|c_{e,l-1}|^2$ in formula (5) by $|c_{e,l-1}|^2 F_{l-1}$ where F_{l-1} is the fidelity defined as $F_{l-1} = 1 - \Gamma(\omega_0) \times \int_0^\tau |c_{e,l-1}(t')|^2 dt'$ from a first order perturbation calculation. This leads to a more precise phonon distribution p_n in the steady state. For parameters corresponding to $\alpha_g \approx 1 \times 10^{-4}$ as given above, the correction to p_n due to the above perturbation effect of the inclusion of CPB dissipation can be omitted when $N_{\rm th} \lesssim 10^2$. On the other hand, the coupling of the CPB to the environment may cause the CPB to jump from the ground state to the excited state with a rate $\Gamma(\omega_0)$, and subsequently emit a phonon via coherent Rabi oscillation. In the low temperature case, this probability can be estimated as $N_{\rm th}\Gamma(\omega_0) \int_0^\tau |C_{e,0}(\tau - t')|^2 dt' = N_{\rm th}\Gamma(\omega_0)\tau/2$. In our example as mentioned above, this value is about $7 \times 10^{-3} N_{\rm th}$. The detailed study on the influence of the dissipation effect of the CPB on our cooling protocol will be given in our future work.

More generally, for the case of finite temperatures, there is always a thermal excitation for the CPB. In fact, even when off duty, the CPB decays to a mixed state with probability $p = 1/(1 + \exp[E_J/k_B T])$ in the excited state $|e\rangle$ and $(1 - p)$ in the ground state $|g\rangle$. For a finite value of p, the elements of the steady state density matrix ρ_s should be modified to be

$$p_n^s = p_0^s \prod_{l=1}^{n} \frac{N_{\rm th} l + p|c_{e,l-1}|^2 r_a/\kappa}{(N_{\rm th} + 1)l + (1 - p)|c_{e,l-1}|^2 r_a/\kappa}. \qquad (7)$$

This leads to a reduction of the cooling efficiency. In Fig. 3, the influence of this thermal excitation probability is illustrated. In the low temperature limit, the average number of phonons emitted by the CPB via this mechanism during the duty cycle is still p. Combined with the analysis of the previous paragraphs, the lower limit phonon number fluctuation at steady state due to thermal excitations and dissipations of the CPB can be estimated as $1/(1 + \exp[E_J/k_B T]) + N_{\rm th}\Gamma(\omega_0)\tau/2$, which vanishes in the limit of large E_J and low $\Gamma(\omega_0)$. This can also be considered as the lower limit of the average phonon number after cooling.

Conclusion with remarks about relations to maser.—We should notice that the periodic cooling of the NAMR can be understood as an "inverse" of the "maser" mechanism. In the usual maser process, the input excited atoms (molecules) can coherently heat the cavity and then *coherently accumulate* photons in a single state to enhance cavity field in a quantum way. In the present protocol, the NAMR is cooled by the CPB in its ground state. It should be emphasized that, with the same setup and operations similar to the above protocol, a "NAMR maser" can be devised if the CPB is initially prepared in its excited state $|e\rangle$ before the interaction is switched on. Namely, we can prepare the NAMR in the nonclassical state with the number of phonons in super-Poissonian or sub-Poissonian distribution. On the other hand, in many protocols of two-qubit quantum logic gates based on different physical systems, the bosonic mode in its ground state can serve as a quantum data bus to transfer quantum information from its coupled qubit to another, or to entangle two qubit at a distance. Therefore, the protocol in this Letter may have potential applications in quantum information theory.

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