Separating the Classical and Quantum Information via Quantum Cloning

M. Ricci,[1] F. Sciarrino,[1] N. J. Cerf,[2] R. Filip,[3] J. Fiurášek,[3] and F. De Martini[1]

[1]*Dipartimento di Fisica and Istituto Nazionale per la Fisica della Materia, Università di Roma "La Sapienza", p.le A. Moro 5, Roma, I-00185, Italy*

[2]*Ecole Polytechnique, Universitė Libre de Bruxelles, Bruxelles, B-1050, Belgium*

[3]*Department of Optics, Palacký University, 17, listopadu 50,Olomouc 77200, Czech Republic*

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An application of quantum cloning to optimally interface a quantum system with a classical observer is presented; in particular, we describe a procedure to perform a minimal disturbance measurement on a single qubit by adopting a $1 \rightarrow 2$ cloning machine followed by a generalized measurement on a single clone and the anticlone or on the two clones. Such a scheme can be applied to enhance the transmission fidelity over a lossy quantum channel.

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Information is a property of physical systems that can be defined and quantified within any physical model. While basic principles are assumed to be generally valid, a coherent analytic formulation of an information theory is deeply related to the strategies by which the knowledge about a system is acquired by a (classical) observer. In quantum theory an observer cannot extract all the information about an unknown state $|\phi\rangle$ by a measurement performed on a finite ensemble of identically prepared systems. In particular, the mean fidelity G of any state estimation strategy based on the measurement of N copies of a qubit $|\phi\rangle$ must satisfy the bound $G \leq G_{\mathrm{opt}} = (N+1)/(N+2)$ [1], where G is defined as the mean overlap between the unknown state $|\phi\rangle$ and the state inferred from the measurement ρ_G: $G = \langle\phi|\rho_G|\phi\rangle$. Moreover, any gain of knowledge irreversibly alters the estimated system. Recently the disturbance associated to the estimation process has been characterized analytically for a generic d-level quantum system and the optimal ratio between the classical information acquired, G, and the quantum fidelity $F = \langle\phi|\rho_S|\phi\rangle$ of the output state ρ_S has been found by Banaszek [2] [Fig. 1(a)]. These fundamental results of the classical-quantum interface theory also affect the quantum process of the distribution of information from a single quantum system to many ones: one of the obvious consequences of the bound on the fidelity of estimation is that unknown states of quantum systems cannot be perfectly copied [3]. Certainly if this would be possible, then one would be able to violate the boundary value G_{opt}. The problem of manipulating and controlling the flux of quantum information has been in general tackled and solved by the theory of quantum cloning [4]. Actually the optimal cloning processes generate copies which exhibit the maximum values of quantum fidelity achievable in compliance to quantum mechanics rules; this feature renders such devices an essential instrument for the assessment of the security of quantum cryptographic protocols [5].

In this work we show that quantum cloning is a fundamental tool not only for the distribution of quantum information but also to interface a quantum system with a classical observer, that is, to optimally split the original information content associated with any system into a classical and a quantum contribution. Precisely, a minimal disturbance measurement on a qubit can be implemented adopting a $1 \rightarrow 2$ universal cloning machine followed by a proper generalized positive operator value measurement (POVM) applied on the two clones or on a single clone and the anticlone. The minimal disturbance implies a measurement that saturates the quantum mechanical trade-off between the information gained by the observer and the quantum state disturbance induced by the estimating process. Two different strategies will be considered: the first one exploits a tunable asymmetric cloning machine followed by a fixed POVM while the second one employs a

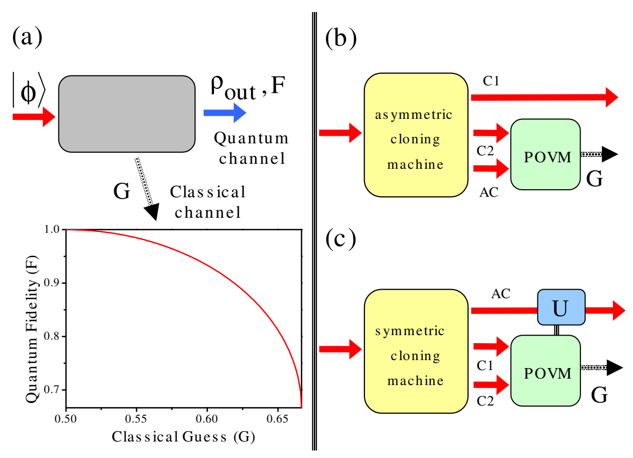

FIG. 1 (color online). Left: plot of the optimal quantum fidelity vs the classical guess of the state. Right: schematic diagrams of a minimal disturbance measurement on a single qubit performed by adopting: (b) an asymmetric cloning machine and a POVM and (c) a symmetric cloning machine, a POVM, and a classical feed-forward.

symmetric cloning machine, a variable POVM, and a classical feed-forward in analogy with the teleportation protocol [6].

Let us consider a single ($N = 1$) qubit, e.g., encoded in the polarization of a single photon. The expression of the Banaszek's bound reads:

$$\sqrt{F - \frac{1}{3}} \leq \sqrt{G - \frac{1}{3}} + \sqrt{\frac{2}{3} - G}. \tag{1}$$

For the sake of simplicity, we restrict our considerations to the $N = 1 \rightarrow M = 2$ asymmetric optimal quantum cloning machine (AQCM) [7,8] which generates two clones $C1$ and $C2$ with different fidelities F_{C1} and F_{C2} starting from the input qubit in the state $|\phi\rangle$ and from two ancillas. For an asymmetric universal cloner, a simple way to express the transformation acting on $|\phi\rangle$ is

$$|\phi\rangle \rightarrow \nu|\phi\rangle_{C1}|\Psi^-\rangle_{C2,AC} + \mu|\phi\rangle_{C2}|\Psi^-\rangle_{C1,AC}, \tag{2}$$

where AC denotes the third qubit, usually called "anticlone" and $|\Psi^-\rangle = 2^{-1/2}(|01\rangle - |10\rangle)$. Here $|\mu|^2$ is the depolarizing fraction of clone $C1$, that is, the probability that $C1$ is depolarized, so that the corresponding fidelity reads $F_{C1}(\mu) = 1 - |\mu|^2/2$. Of course, $|\nu|^2$ is the depolarizing fraction of clone $C2$, and we have a similar expression $F_{C2}(\nu) = 1 - |\nu|^2/2$. The normalization condition reads $|\mu|^2 + \mu\nu + |\nu|^2 = 1$. Note that ($\mu = 0$; $\nu = 1$) is a trivial cloner where the original is transferred to $C1$, while $C2$ is random. The value $\mu = \nu = 1/\sqrt{3}$ corresponds to the symmetric machine, hence $F_{C1} = F_{C2} = 5/6$. The cloning transformation (2) can be reexpressed in the following form

$$|\phi\rangle \rightarrow \left(\frac{\mu}{2} + \nu\right)\{|\phi\rangle_{C1}|\Psi^-\rangle_{C2,AC}\} + \frac{\mu}{2}\{\sigma_Z|\phi\rangle_{C1}|\Psi^+\rangle_{C2,AC} + \sigma_X|\phi\rangle_{C1}|\Phi^-\rangle_{C2,AC} + \sigma_Y|\phi\rangle_{C1}|\Phi^+\rangle_{C2,AC}\}, \tag{3}$$

where $|\Psi^\pm\rangle = 2^{-1/2}(|01\rangle \pm |10\rangle)$ and $|\Phi^\pm\rangle = 2^{-1/2}(|00\rangle \pm |11\rangle)$ are the four Bell states of the qubits $C2$ and AC. A straightforward conclusion we can draw from this expression is that performing a Bell measurement on $C2$ and AC gives all the information needed to perfectly reconstruct the original state $|\phi\rangle$ from $C1$ [9]. Just like in teleportation, we need to apply one of the Pauli operators on $C1$ depending on the outcome of the Bell measurement. Also, by tracing over $C2$ and AC, we see that the clone $C1$ is left in the state $\rho_{C1} = (1 - |\mu|^2)|\phi\rangle\langle\phi| + |\mu|^2\mathbb{I}/2$. Similar conclusions can be obtained for the clone $C2$ and the anticlone AC, which is in state $\mu\nu|\phi_\perp\rangle\langle\phi_\perp| + (|\mu|^2 + |\nu|^2)\mathbb{I}/2$, where $|\phi_\perp\rangle$ denotes a state orthogonal to $|\phi\rangle$, $\langle\phi_\perp|\phi\rangle = 0$.

Asymmetric cloning.—The basic idea of the present scheme is the following: the quantum information carried by the input system $|\phi\rangle$ is distributed into a larger number of qubits adopting the AQCM and then, while the clone $C1$ contains an approximate replica of $|\phi\rangle$ quantified by the fidelity $F_{C1} = F$, the other outputs of the machine, $C2$ and AC, are coherently measured to acquire classical information on the initial state and estimate it with fidelity G [Fig. 1(b)]. The information preserving property of the cloning process suggests that the optimal value of G achievable by this procedure should satisfy the Banaszek's bound for the given value of F. Intuitively the optimal procedure could consist of the state estimation of a state $|\phi\rangle$ from a pair of orthogonal qubits $|\phi\rangle|\phi_\perp\rangle$.

Let us first establish the formalism for the POVM. The optimal covariant POVM for the estimation of $|\phi\rangle$ from a single copy of $|\phi\rangle \otimes |\phi_\perp\rangle$ has the structure

$$\Pi(\Omega) = U(\Omega) \otimes U(\Omega)\Pi_0 U^\dagger(\Omega) \otimes U^\dagger(\Omega), \tag{4}$$

where the unitary $U(\Omega)$ generates the states $|\Omega\rangle$ and $|\Omega_\perp\rangle$ from the computational basis states, $|\Omega\rangle = U(\Omega)|0\rangle$ and $|\Omega_\perp\rangle = U(\Omega)|1\rangle$. The POVM $\Pi(\Omega)$ must satisfy the normalization condition, $\int_\Omega \Pi(\Omega)d\Omega = \mathbb{I}$, where $\mathbb{I}$ denotes the identity operator and $d\Omega$ is the invariant Haar measure on the group SU(2). If the successful measurement is $\Pi(\Omega)$, then the estimated state reads $U(\Omega)|0\rangle$. The operator Π_0 which generates the optimal POVM (4) has rank one and is found to be $\Pi_0 = |\pi_0\rangle\langle\pi_0|$, where $|\pi_0\rangle = \frac{\sqrt{3}+1}{\sqrt{2}}[|01\rangle + (2 - \sqrt{3})|10\rangle]$. The covariant POVM is continuous but it can be discretized by choosing only several particular $\Omega_j \equiv (\vartheta_j, \phi_j)$ such that $\Sigma_j \Pi(\Omega_j) \propto \mathbb{I}$. As found by [10] an optimal strategy consists in the following discrete POVM $|\Theta_i\rangle\langle\Theta_i|$, $\{i = 1, 4\}$

$$|\Theta_i\rangle = \gamma|\vec{n}_i, -\vec{n}_i\rangle - \delta\sum_{k\neq i}|\vec{n}_k, -\vec{n}_k\rangle,$$

where $\gamma = 13/(6\sqrt{6} - 2\sqrt{2})$, $\delta = (5 - 2\sqrt{3})/(6\sqrt{6} - 2\sqrt{2})$, $\{\vec{n}_i\}$ represents the directions of the four vertices of a tethraedron in the Bloch sphere with the following Cartesian coordinates $\vec{n}_1 = (0, 0, 1)$, $\vec{n}_2 = (\frac{\sqrt{8}}{3}, 0, -\frac{1}{3})$, $\vec{n}_3 = (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3})$, $\vec{n}_4 = (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3})$.

The initial state of the quantum system $C2$-AC in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$ is expressed by the density matrix $\rho_{C2,AC}$, attained by tracing over the system $C1$ in Eq. (2). Applying the POVM $|\Theta_i\rangle\langle\Theta_i|$ the output i is obtained with probability $p_i = \mathrm{tr}(|\Theta_i\rangle\langle\Theta_i|\rho_{C2,AC})$ and the input qubit is guessed to be in the state $|\vec{n}_i\rangle$. The amount of classical information about $|\phi\rangle$ attained is $G(|\phi\rangle) = \Sigma_i p_i |\langle\phi|\vec{n}_i\rangle|^2$ and the average value of G over all possible input states is equal to $G = \int_{|\phi\rangle\epsilon H} G(|\phi\rangle)d\phi$. From the previous expressions we obtain for F and G the functional relation

$$F(G) = \frac{1}{3} + (\sqrt{G - 1/3} + \sqrt{-G + 2/3})^2 \tag{5}$$

that saturates the Banaszek's bound (1). Note that the optimal measurement on the second clone and anticlone does not depend on the asymmetry of the cloner. The latter is only used here to tune the balance between F and G.

Symmetric cloning.—Let us now investigate whether the optimal trade-off between F and G can be obtained by varying the measurement on the two clones generated through the symmetric cloning machine. In this case the optimal covariant POVM is generated by the rank-one operator $\tilde{\Pi}_0 = |\tilde{\pi}_0\rangle\langle\tilde{\pi}_0|$, where $|\tilde{\pi}_0\rangle = \xi|00\rangle + \sqrt{3-\xi^2}|11\rangle$. This measurement interpolates between optimal POVM for state estimation from $|\phi\rangle|\phi\rangle$ ($\xi = \sqrt{3}$) leading to the maximum value $G_{\text{opt}} = \frac{2}{3}$ and the Bell measurement in the basis of maximally entangled states ($\xi = \sqrt{3/2}$) leading to $F = 1$ through a reversion strategy.

Let us calculate the estimation fidelity for the covariant POVM generated by $\tilde{\Pi}_0$. The mean fidelity can be calculated by averaging over all input states and over the POVM. However, since the POVM is covariant, it suffices to consider only a single input state, e.g., $|0\rangle$. By exploiting the expression (2) for $\mu = 1/\sqrt{3}$ and $|\phi\rangle = |0\rangle$ we obtain the average fidelity $G = \int_\Omega \text{tr}(\tilde{\Pi}(\Omega)\rho_{C1,C2})|\langle 0|U(\Omega)|0\rangle|^2 d\Omega$, where $\rho_{C1,C2}$ is the reduced density matrix of systems $C1 - C2$. The final result is $G = \frac{1}{3} + \frac{\xi^2}{9}$. If the measurement result is $\tilde{\Pi}(\Omega)$, then the correcting unitary $U(\Omega)U^T(\Omega)\sigma_Y$ should be applied to the anticlone C. The mean fidelity between the anticlone after this correction and the input state $|\phi\rangle$ can be evaluated as $F = \frac{2}{3} + \frac{2}{9}\xi\sqrt{3-\xi^2}$. If we express ξ in terms of G, we find that $F(G)$ is equal to expression (5). This proves that the Banaszek's bound is saturated. We have assumed here that the optimal POVM is covariant and continuous but we could of course discretize it and find an equivalent POVM with finite number of elements.

Applications.—In the present paragraphs, we shall exploit the quantum cloning to improve a simple quantum communication task. Let us consider the following problem: Alice wants to transmit an unknown quantum state $|\phi\rangle$ encoded into a single photon to Bob through a lossy channel [Fig. 2(a)]. The quantum communication channel is characterized by the transmittivity p, i.e., the probability that the photon reaches Bob's station. In the case in which Alice directly sends the photon to Bob, the fidelity of the quantum state transmission is found to be $F_{\text{dir}} = (1 + p)/2$ [Fig. 2(b)]. Indeed when the qubit reaches Bob, an event occurring with probability p, the fidelity of transmission is equal to 1. Otherwise, when the qubit is lost, Bob must guess randomly the quantum state of $|\phi\rangle$ and the fidelity is equal to $\frac{1}{2}$. In order to enhance this transmission fidelity we shall investigate different alternative strategies based on the cloning process.

In a first scenario involving the asymmetric cloner, the clone $C1$ is sent down the quantum channel while the qubits $C2$ and AC are kept at the sender station [Fig. 1(b)]. Alice optimally estimates the input state with fidelity G by performing the POVM $\Pi(\Omega)$ on $C2$ and AC [Eq. (4)] and communicates the result to Bob. Let us first note that if no memory is available and the measurement on sender's side

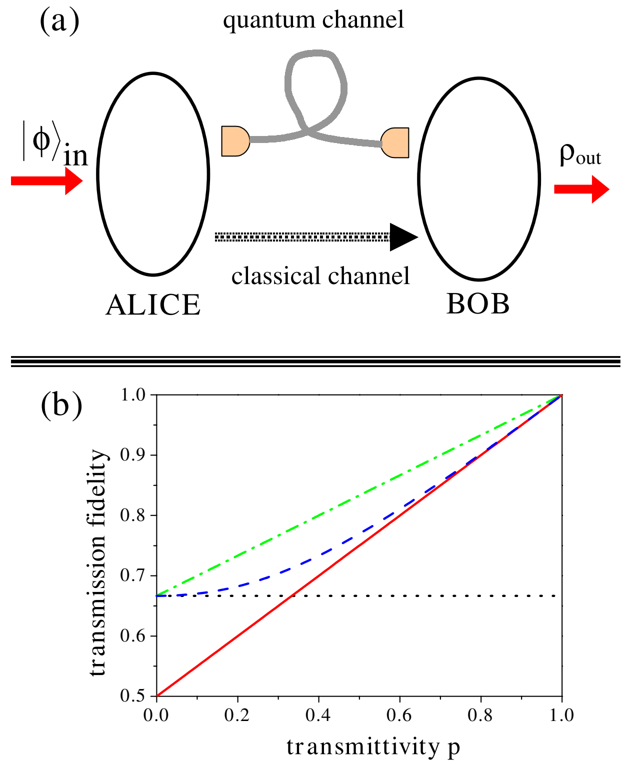

FIG. 2 (color online). (a) Schematic diagram of a generic communication channel; (b) fidelity of the transmission of a quantum state through a quantum channel characterized by a transmittivity p: solid line (F_{dir}), dashed line (F_{cl}), dotted line (G_{opt}), and dash-dotted line (F_{QM}).

is independent of whether the state was delivered to receiver or lost in the channel, then the Banaszek's bound applies and cannot be beaten. The overall transmission fidelity is now F_{C1} when the qubit reaches Bob and G when Bob is forced to exploit the classical information, since the photon is lost. Hence the average fidelity reads $F(p) = pF_{C1} + (1 - p)G$. By optimizing the asymmetry of the cloning machine, that is, the parameter μ, with respect to the transmittivity p of the channel we obtain [Fig. 2(b)] $F_{\text{cl}}(p) = \frac{1}{6}(3 + p + \sqrt{1 + p(5p - 2)})$. This is the optimal strategy based on a classical-quantum communication since the present procedure saturates the Banaszek's bound, as said. Similar results can be obtained by adopting a symmetric cloning machine at the sender station. In this case the POVM $\tilde{\Pi}(\Omega)$ is performed on the two clones [Fig. 1(c)]: depending on the result Alice applies the appropriate feed-forward U to the AC qubit and then sends it to Bob.

A higher fidelity of transmission can be obtained by a more sophisticated approach. Let us suppose that Alice can use a quantum memory [11] whereas Bob can communicate to her whether or not he has received the transmitted photon. If the photon reaches Bob's site, they apply a reversion procedure and recover the initial qubit $|\phi\rangle$ at Bob's station. Two different strategies which lead to the same fidelity of transmission [Fig. 2(b)] $F_{\rm qm} = \frac{2}{3} + \frac{1}{3}p$ are possible. In the first approach Alice employs a symmetric cloning and transmits the anticlone to Bob. If the photon is lost, Alice performs an optimal estimation on the clones achieving a fidelity $\frac{2}{3}$, otherwise she carries out an incomplete Bell measurement on the two clones and sends her results to Bob that applies the appropriate unitary Pauli operator to the qubit AC in order to recover $|\phi\rangle$. Since the two clones belong to the symmetric subspace 1 trit of information must be transmitted from Alice to Bob. The quantum memory is necessary since Alice must wait for Bob's message to decide whether she should implement a Bell measurement or an estimation POVM. The second approach, based on the AQCM, reduces to the standard teleportation protocol over a lossy quantum channel with transmission probability p.

The quantum cloning can also be used to protect from losses a state stored in a quantum memory. Consider a simple model where a qubit stored in a memory is preserved with probability p and is erased with probability $1 - p$ leading to a fidelity of storage $F_S = (1 + p)/2$. Suppose now that before storing we clone the state and keep in the memory both clones as well as the anticlone. If all three qubits are preserved, we perfectly recover the state; otherwise, if at least one clone is maintained, we get the fidelity $5/6$. If only the anticlone is preserved, then we can apply another approximate U-NOT gate and recover a state with fidelity $5/9$. Finally, when all qubits are lost we guess the state with fidelity $1/2$. The average fidelity of this cloning-based strategy reads $F_C = (1 + 2p) \times (9 - 5p + 2p^2)/18$. Remarkably, $F_C - F_S$ is non-negative for all $p \in [0, 1]$ so the cloning can partially protect the state in the memory from the erasure. The improvement is maximum for $p = 1/3$ when we obtain $F_C - F_S = 3.3\%$.

Conclusions.—We have presented an explicit application of quantum cloning to quantum-classical interface; in particular, we described a procedure to perform a minimal disturbance measurement. Such a scheme can been applied to enhance the transmission fidelity over a lossy (but noiseless) channel and the performance of a quantum memory with erasure. These procedures exploit the cloning process to encode a single qubit into the Hilbert space of three qubits. This redundancy, similar to the one exploited in quantum error correction, is the reason why the cloning can help in protecting quantum information from losses. We may expect that the cloning can also help to protect against other kinds of decoherence, even if theoretical analysis reveals that the present strategy does not work for depolarizing channels. As a further application, we note that the present method realizes a universal weak measurement [12] in the limit of vanishing disturbance.

The implementation of the previous schemes adopting single photon states is challenging but within the present technology. The cloning machine has been realized either by an amplification process [8,13,14] and by linear optics techniques [8,15,16]; classical feed-forward has also recently been reported [17], and the required generalized measurements (POVMs) on two photonic qubits can be realized probabilistically using linear optics.

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