Quasiregular Concentric Waves in Heterogeneous Lattices of Coupled Oscillators

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We study the pattern formation in a lattice of locally coupled phase oscillators with quenched disorder. In the synchronized regime quasiregular concentric waves can arise which are induced by the disorder of the system. Maximal regularity is found at the edge of the synchronization regime. The emergence of the concentric waves is related to the symmetry breaking of the interaction function. An explanation of the numerically observed phenomena is given in a one-dimensional chain of coupled phase oscillators. Scaling properties, describing the target patterns are obtained.

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The study of coupled oscillators is one of the fundamental problems in theoretical physics and has led to many insights into the mechanisms of spatiotemporal pattern formation in oscillatory media [1–4]. Renewed interest stems from its possible role in many biological systems like cardiac tissue [5], neural [6] and ecological systems [7]. However, nearly all theoretical studies have been done with idealized systems of identical oscillators, whereas not much is known about the dynamics and pattern formation in heterogeneous media [8].

Target waves are one of the most prominent patterns in oscillatory media and are usually associated with the presence of local impurities in the system [1,9,10]. These pacemakers change the local oscillation frequency and are able to enslave all other oscillators in the medium, which finally results in regular ring waves [1,11]. However, the assumption of a discrete set of localized pacemaker regions in an otherwise homogeneous medium is somewhat artificial. Especially biological systems are often under the constraint of large heterogeneity. In such a disordered system no point can clearly be distinguished as a pacemaker and it is not obvious whether such a system can sustain highly regular target patterns and where they should originate.

The emergence of target patterns in heterogeneous oscillatory media was first reported and explained in [12] and subsequently observed in [7]. In this Letter we show that the random nature of the medium itself plays a key role in the formation of the patterns. As the disorder in a rather homogeneous synchronized medium is increased we observe the formation of quasiregular target waves, which result from an intricate interplay between the heterogeneity and a symmetry breaking of the coupling function. Further, we obtain the scaling properties of the synchronization frequency and wavelength in the limit of small phase differences between neighboring oscillators.

We study a system of N coupled phase oscillators [1]

$$\dot{\theta}_i = \omega_i + \epsilon \sum_{j\in N_i} \Gamma(\theta_j - \theta_i), \qquad i = 1, \ldots, N. \qquad (1)$$

Here, θ_i represents the phase of oscillator i, which is coupled with strength ϵ to a set of nearest neighbors N_i in a one- or two-dimensional lattice. The natural frequencies ω_i are fixed in time, uncorrelated, and taken from a distribution $\rho(\omega)$. A scaling of time and a transformation into a rotating reference frame can always be applied so that $\epsilon = 1$ and the ensemble mean frequency $\bar{\omega}$ is equal to zero. We refer to the variance $\sigma^2 = \mathbf{var}(\omega_i)$ of the random frequencies as the disorder of the medium.

The effects of coupling are represented by an interaction function Γ which, in general, is a 2π-periodic function of the phase difference with $\Gamma(0) = 0$. For weakly coupled, weakly nonlinear oscillators, Γ has the universal form [1,13]

$$\Gamma(\phi) = (\sin(\phi) + \gamma[1 - \cos(\phi)]). \qquad (2)$$

The symmetry breaking parameter γ describes the nonisochronicity of the oscillations [1].

It is well known that for sufficiently small disorder the oscillators eventually become entrained to a common locking frequency Ω [1–3]. Since the oscillators are nonidentical, even in this synchronized state they are usually separated by fixed phase differences. These can sum up over the whole lattice to produce spatiotemporal patterns, which are characterized by a stationary phase profile, $\theta_i - \theta_1$. This is demonstrated in Fig. 1, where we have simulated system (1) with interaction (2) in a two-dimensional lattice. If the heterogeneity is small, the oscillations across the lattice synchronize to a homogenous phase profile [Fig. 1(a)]. However, by increasing the disorder we observe the formation of target waves with decreasing wavelength [Figs. 1(a)–1(c)]. The emergence of the concentric waves is due to the symmetry breaking in the interaction function. For example, by reducing γ in Eq. (2) the pattern becomes more irregular [Fig. 1(d)].

It is the counterintuitive observation that the isotropic medium with random frequency distributions of no spatial correlation (see Fig. 1 insets) can generate and sustain very regular wave patterns. Since the Eqs. (1) generally approximates the phase dynamics for coupled limit cycle oscil-

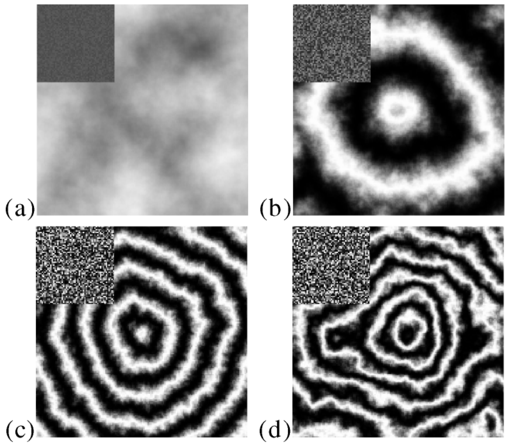

FIG. 1. Simulation results in a two-dimensional lattice of 150x150 phase oscillators (1) and (2) with nearest neighbor coupling, $\epsilon = 1$, periodic boundary conditions, homogeneous initial conditions, and (a) $\gamma = 2$, $\sigma = 0.029$, (b) $\gamma = 2$, $\sigma = 0.173$, (c) $\gamma = 2$, $\sigma = 0.433$, and (d) $\gamma = 0.3$, $\sigma = 0.433$. The random frequencies are taken from a uniform distribution of variance σ^2. Plotted is the sine of the phases θ_i as gray level. Similar results are obtained for open boundaries. (a)–(c): effect of increased heterogeneity; (c)–(d): influence of nonisochronicity γ. Insets show the natural frequencies ω_i as gray levels.

lators this effect is not restricted to phase equations (1). Similar disorder induced target patterns have been observed in lattices of a variety of oscillator types, including predator-prey and neural systems, chemical reactions, and even chaotic oscillators [7,12].

In the synchronized state, all oscillators rotate with the constant locking frequency $\dot{\theta}_i = \Omega$, so that system (1) becomes a set of N equations, which have to be solved self consistently for the phases θ_i and Ω under some imposed boundary conditions. To determine Ω, suppose first that the coupling function Γ is fully antisymmetric $\Gamma(-\phi) = -\Gamma(\phi)$, e.g., $\gamma = 0$ in Eq. (2). In this case, by summing up all equations in (1) we obtain $\Omega = \bar{\omega} = 0$ in the rotating frame. Thus, nontrivial locking frequencies $\Omega \neq 0$ only arise if $\Gamma(\phi)$ has a symmetric part $\Gamma_S(\phi) = \frac{1}{2}[\Gamma(\phi) + \Gamma(-\phi)]$,

$$\Omega = \frac{1}{N} \sum_{i,j \in N_i} \Gamma_S(\theta_j - \theta_i). \tag{3}$$

For any coupling function Γ, given a realization of the natural frequencies ω_i, we ask for the resulting phase profile θ_i. Note that the inverse problem is easy to solve: for any regular phase profile θ_i we can calculate Ω from Eq. (3), which after inserting into Eq. (1), yields the frequencies ω_i.

Insights into the pattern formation can be gained from a one-dimensional chain of phase oscillators [12,14]

$$\Omega = \omega_i + [\Gamma(\phi_i) + \Gamma(-\phi_{i-1})]. \tag{4}$$

Here we use $\phi_i = \theta_{i+1} - \theta_i$ for the phase differences between neighboring oscillators, $i = 1, \ldots, N-1$. Further, we assume open boundary conditions $\phi_0 = \phi_N = 0$. The self consistency problem is trivial for an antisymmetric Γ where $\Omega = 0$. In this case the $\Gamma(\phi_i)$ simply describe a random walk $\Gamma(\phi_i) = -\Sigma_{j=1}^{i} \omega_j$. Thus, for small $|\phi_i|$ the phase profile θ_i is essentially given by a double summation, i.e., a smoothening, over the disorder ω_i [see Figs. 2(a) and 2(b)]. Note that synchronization can only be achieved as long as the random walk stays within the range of Γ. Thus, with increasing system size N, synchronization becomes more and more unlikely.

The emergence of target waves is connected to a breaking of the coupling symmetry. To explore this we study a unidirectional coupling with respect to ϕ,

$$\Gamma(\phi) = f(\phi)\Theta(\phi), \quad \text{for} \quad |\phi| \ll 1, \tag{5}$$

with the Heaviside function $\Theta(\phi)$ and $f(\phi > 0) > 0$. Here, the phase of oscillator i is only influenced from neighboring oscillators which are ahead of i. If the solution is small phase differences we are not concerned about the periodicity of $\Gamma(\phi)$ as the coupling is only required to be unidirectional close to zero. For open boundaries $\phi_0 = \phi_N = 0$ the solution to (4) and (5) is given by

$$\phi_i = \begin{cases} f^{-1}(\Omega - \omega_i), & i < m \\ -f^{-1}(\Omega - \omega_{i+1}), & i \geq m \end{cases}. \tag{6}$$

Here, the index m is the location of the oscillator with the largest natural frequency, which also sets the synchronization frequency

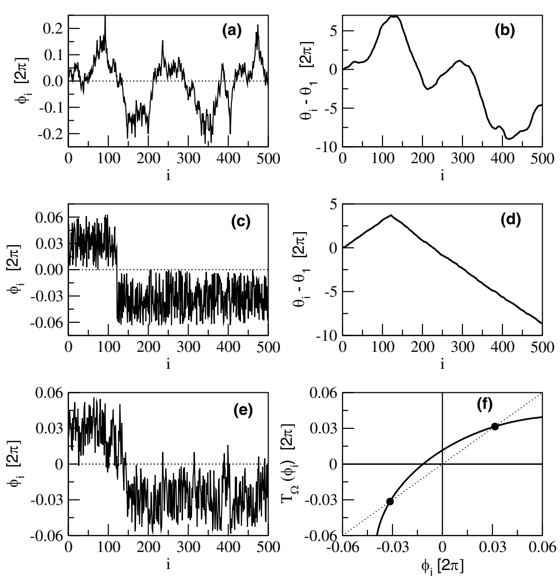

FIG. 2. Phase differences ϕ_i (left) and phase profile $\theta_i - \theta_1$ (b),(d) in units of 2π for a chain of 500 phase oscillators (4) with uniformly distributed frequencies $\omega_i \in [-0.1, 0.1]$ and open boundaries. (a),(b) antisymmetric coupling (2) with $\gamma = 0$. (c),(d) unidirectional coupling (5) with $\Gamma(\phi) = \phi\Theta(\phi)$. (e) System (2) with $\gamma = 2$. (f) Transfer map $T_\Omega(\phi_i)$ (8) for Eq. (2) with $\gamma = 2$ (solid line) and fixed points ϕ^* (solid circles).

$$\Omega = \omega_m = \mathbf{max}_i(\omega_i). \tag{7}$$

The phase differences (6) are positive to the left of the fastest oscillator, $i < m$, and negative to the right $i \geq m$. As a consequence, the phase profile has a tent shape with a mean slope that is given by averaging (6) with respect to the frequency distribution [Figs. 2(c) and 2(d)]. We call this solution type a quasiregular concentric wave. This example illustrates that the asymmetry of the coupling function increases the influence of faster oscillators and effectively creates pacemakers with the potential to entrain the whole system. Note, that the solution (6) and (7) is not possible without disorder, i.e., for $\sigma = 0$. Further, in contrast to the antisymmetric coupling, here synchronization can be achieved for chains of arbitrary length.

In general, the coupling function Γ will interpolate between the two extremes of fully antisymmetry and unidirectional coupling in the vicinity of zero, e.g., Eq. (2) with $\gamma \neq 0$. As shown in Fig. 2(e) this also gives rise to quasiregular concentric waves, very similar to the exactly solvable system Fig. 2(c). To further investigate the origin of these patterns, note that for any given Ω system (4) implicitly defines two transfer maps, $T_\Omega:\{\phi_{i-1}, \omega_i\} \mapsto \phi_i$ and $T_\Omega^{-1}:\{\phi_i, \omega_i\} \mapsto \phi_{i-1}$, which describe the evolution of the phase differences into the right or the left direction of the chain, respectively. The random frequencies ω_i can be seen as noise acting on the map [see Fig. 2(f)]

$$\phi_i = T_\Omega(\phi_{i-1}, \omega_i) = \Gamma^{-1}(\Omega - \omega_i - \Gamma(-\phi_{i-1})). \tag{8}$$

It is easy to see that the breaking of symmetry leads to a pair of fixed points, ϕ^* and $-\phi^*$, in the noisy maps

$$\phi^* = T_\Omega(\phi^*, \bar{\omega}) = \Gamma_S^{-1}\left(\frac{\Omega}{2}\right). \tag{9}$$

The transfer map can be linearized at the fixed points so that $T_\Omega(\phi^* + \zeta, \omega) \approx \phi^* + a\zeta - b\omega$ with $a = \frac{\Gamma'(-\phi^*)}{\Gamma'(\phi^*)}$ and $b = \frac{1}{\Gamma'(\phi^*)}$. While one fixed point, ϕ^* in the case (2) with $\gamma > 0$, is linearly stable ($|a| \leq 1$) the other fixed point is necessarily unstable. These stability properties are inverted for T_Ω^{-1}. Thus, when iterating to the right of the chain the ϕ_i are concentrated around ϕ^* and around $-\phi^*$ when iterating to the left. As a consequence, the general solution of the self consistency problem (4) is built up from two branches around the two fixed points $\pm\phi^*$, superimposed by autocorrelated fluctuations ζ_i (see Fig. 2)

$$\phi_i = \pm\phi^* + \zeta_i. \tag{10}$$

After summation, this leads to the quasiregular tent shape of the phase profile θ_i. In a first approximation the ζ_i describe a linear autoregressive stochastic process

$$\zeta_i = a\zeta_{i-1} - b\omega_i. \tag{11}$$

We want to stress that the fluctuations ζ_i are an essential ingredient of the solution. Although the emerging concentric waves seem to be regular, the underlying heterogeneity of the system does not permit analytical traveling wave solutions $\theta_i(t) = \Omega t - k|i - m|$.

The general solution (10) allows for very different phase profiles (see Fig. 2). The regularity of the wave pattern depends on the relative influence of the mean slope ϕ^* compared to the fluctuations ζ_i and can be measured by the quality factor $Q_\phi = \phi^{*2}/\mathbf{var}(|\phi_i|)$ and the autocorrelation r of the ϕ_i. As demonstrated in Fig. 3, for sufficiently large systems, both Q_ϕ and r only depend on the product $\gamma\sigma$ (see below). For $\gamma\sigma \to 0$ we find $Q_\phi \to 0$ and $r \to 1$, and the solution is essentially a random walk [see Figs. 2(a) and 2(b)]. With increasing values of $\gamma\sigma$ the correlations r are reduced and eventually become negative. Furthermore, Q_ϕ increases with the product $\gamma\sigma$, and for $\gamma > 1$ can rise drastically [see Fig. 3(c)]. Thus, with increasing disorder of the system we obtain more regular patterns until synchronization is lost.

A straightforward integration of system (1) can be problematic due to the long transients. Another approach, which also applies for two-dimensional lattices, relies on the Cole-Hopf transformation of system (1). Assume that the ϕ_i are small so that it is possible to approximate the

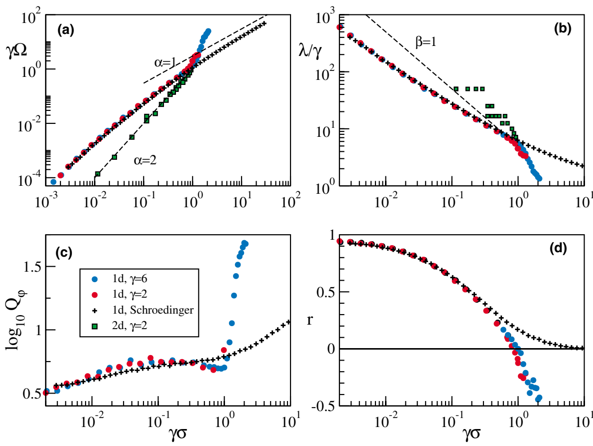

FIG. 3 (color). Characterization of wave patterns by the locking frequency $\gamma\Omega$ (a), the wave length λ/γ (b), the quality factor Q_ϕ (c), and the cross correlation r between neighboring phase differences (d). Numerical solutions are obtained by integrating system (1) and (2) for one-dimensional (circles) and two-dimensional lattices (squares) in the synchronization regime. In one dimension the integrations were carried out with chains of length 500 and averaged over 50 simulation runs for $\gamma = 2$ (red circles) and $\gamma = 6$ (blue circles). In the two-dimensional system each point (green squares) represents one single simulation in a 100x100 lattice with $\gamma = 2$. The results using the eigenvector method (12) are shown as (black $+$). Each point represents an average of 500 simulations with $N = 256$. Further, indicated in (a),(b) are straight lines with a given exponent α and β (dashed lines). The wavelength was obtained for one dimension as $\lambda = 2\pi/\phi^*$ and in the two-dimensional system from a Fourier analysis of the phase profile. The plateaus in the wavelength plot are finite size effects.

coupling function (2) around zero by $\Gamma(\phi) = \frac{1}{\gamma} \times (e^{\gamma\phi} - 1) + O(\phi^3)$. After the Cole-Hopf transformation $\theta_i = \frac{1}{\gamma} \ln q_i$, the synchronized lattice (1) is reduced to a linear system [1,10,12],

$$\dot{q}_i = Eq_i = \gamma\sigma\eta_i\, q_i + \sum_{j\in N_i} (q_j - q_i), \qquad (12)$$

where the random frequencies $\eta_i = \omega_i/\sigma$ are of zero mean and variance one and $E = \gamma\Omega$ is some eigenvalue. System (12) is known as the tight binding model for a particle in a random potential on a lattice [15]. The eigenvector $\mathbf{q}^{\max}$ corresponding to the largest eigenvalue $E_{\max}$ will, in the retransformed system of angles, outgrow the contribution of all other eigenvectors to the time dependent solution linearly in time. If the largest eigenvalue is nondegenerate, the unique synchronized solution is

$$\theta_i(t) - \theta_1(t) = \frac{1}{\gamma} \log\left(\frac{q_i^{\max}}{q_1^{\max}}\right). \qquad (13)$$

Equation (13) is well defined since the components of $\mathbf{q}^{\max}$ do not change sign. Anderson localization theory [15] predicts exponentially decaying localized states with some localization length l, which after applying the reverse Cole-Hopf transformation yields the observed tent-shape phase profile with wavelength $\lambda \sim \gamma l$. Concentric waves emerge when λ becomes smaller than the system size.

For extremal values of $\gamma\sigma$, the system (12) has well defined scaling properties $E_{\max} \sim (\gamma\sigma)^\alpha$ and $l \sim (\gamma\sigma)^{-\beta}$ [15]. Perturbation theory yields $\alpha = 2$ for $\gamma\sigma \ll 1$ and $\alpha = 1$ for $\gamma\sigma \gg 1$. For the exponent β we find $\beta \lesssim 1$ in the one-dimensional system, while $1 \leq \beta \leq 2$ in the two-dimensional lattice. This implies for the synchronization frequency Ω and the wavelength λ

$$\Omega \sim \gamma^{\alpha-1}\sigma^\alpha, \qquad \lambda \sim \gamma^{1-\beta}\sigma^{-\beta}. \qquad (14)$$

Here, γ does not influence the wavelength as much as σ but while an increase of γ in one dimension leads also to an increasing wavelength, the effect in two dimensions is the opposite.

In one dimension the scaling with $\gamma\sigma$ holds as long as Eq. (12) can approximate the transfer map reasonably. The approximation breaks down when the correlation r of the phase differences becomes negative. In this regime the quality factor Q_ϕ strongly depends on both $\gamma\sigma$ and γ. The noise term $b\omega_i$ in (11) can become small with increasing γ. This regime, which is not described by the Anderson approximation, can produce very regular concentric waves near the border of desynchronization. Further, our observables (Fig. 3) only depend on the system size for $N \lesssim 100$.

The constructive role of noise has often been studied [16]. It has been shown that in excitable systems spatial noise can enhance the pattern formation and, for example, is able to promote traveling waves [17]. In heterogeneous oscillatory media the emergence of target patterns was reported in [12]. Here, we have analyzed the origin of these structures and, in particular, we have shown that fluctuations are essential for the observed dynamics. Whereas local coupling tends to synchronize the oscillators, the imposed disorder tends to desynchronize the array. The tension between these two opposing forces can give rise to quasiregular target patterns.

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