Giant Excess Noise and Transient Gain in Misaligned Laser Cavities

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The excess noise factor is calculated analytically for a very general class of optical cavities, and is shown to have a superexponential dependence on cavity misalignment, easily attaining values of order 10^{10}. The physical basis is shown to be "transient gain" associated with amplified spontaneous emission. Similarly dramatic effects of symmetry breaking can be expected in other physical systems with non-normal modes.

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It has been known since about 1979 [1] that lasers with a transverse gain or loss profile show "excess noise," in that the measured noise in the fundamental mode exceeds the one photon per mode expected from fundamental theory [2] by what is now usually known as the Petermann factor, K. This has been shown to be due to the fact that the modes of such lasers are not *power orthogonal* [3,4]. The effect has been observed and thoroughly investigated in unstable resonators [3,5–7], and analogous behavior has been predicted and/or observed in other systems with non-normal modes, including the polarization [8] and time [9,10] domains.

In aligned resonators usually $K < 10^4$ [3–5]. In this Letter we show that in misaligned optical systems $K \sim \exp(\text{constant} \times \text{misalignment})^2$, easily leading to $K = 10^{10}$ or more before lasing is killed. Such giant values cannot simply be interpreted as "leakage" of multimode noise into the lasing mode. We show that the maximum gain of a perturbation as it evolves towards the (stable) lasing mode is exactly equal to K, leading to a physical interpretation of this extreme sensitivity to noise in terms of contamination of the lasing mode by strongly amplified spontaneous emission; cf. [4]. It is unusual and remarkable to find such huge numbers for generic parameters, i.e., unconnected with any singularity. That we find such behavior in an exactly soluble model is particularly interesting, because it enables its physical origin and parameter dependences to be studied. Our approach is applicable to a wide class of misaligned cavities, and can be further generalized. We expect these ideas to have even wider applicability, given the many fields in which non-normal modes are important [11,12].

As mentioned, the transverse modes of laser cavities are not, in general, power orthogonal. Instead, there exists a set of *adjoint modes,* which are *biorthogonal* to the cavity modes, and thus act as projectors onto the modes of, e.g., noise. In the simplest case (which is actually much more general than it appears, as we show), the laser modes $U_i(x)$ and their adjoints are identical. They obey

$$\int_{-\infty}^{\infty} dx U_i U_j = 0 \qquad i \neq j. \tag{1}$$

(We consider one transverse dimension, and generalization is straightforward.) For this case, setting $U(x)$ as the fundamental ($i = 0$) mode, its Petermann factor has a particularly simple form:

$$K = \frac{(\int_{-\infty}^{\infty} dx|U|^2)^2}{|\int_{-\infty}^{\infty} dx U^2|^2}. \tag{2}$$

Suppose now that U is a complex Gaussian beam with an offset h and tilt μ:

$$U(x) = e^{-(X+iY)(x-h)^2/2+i\mu x}. \tag{3}$$

(Beam confinement requires $X > 0$.) For such a mode,

$$K = \frac{\sqrt{X^2+Y^2}}{X} \exp\left[\frac{2X\mu^2}{(X^2+Y^2)}\right]. \tag{4}$$

The exponential clearly arises from misalignment, while its coefficient is also present for the aligned cavity, so we write $K = K^{\rm al}K^{\rm mis}$ [13]. $K^{\rm al}$ evidently increases with wave front curvature, but only algebraically. In contrast, $K^{\rm mis}$ has a superexponential dependence on the tilt parameter μ, which can lead to giant excess noise even for modest parameter values. The reason is clear from (2): for finite Y (curved wave front), the integrand U^2 in the denominator oscillates, leading to some cancellation; hence $K^{\rm al} > 1$. However, the frequency of these oscillations vanishes at $x = h$, where $|U|$ is maximal, so that the effect on K is fairly weak. For $\mu \neq 0$, however, U^2 has a finite oscillation frequency at $x = h$, leading to efficient cancellation and strong enhancement of K. We now show that the above analysis can be generalized to the widest class of laser cavity for which analytic mode functions are available.

The modal properties of aligned laser cavities are normally found by using the $ABCD$ matrix approach to beam propagation around the cavity [3]. This approach can be extended to embrace cavities with "soft" apertures and mirrors (with a Gaussian attenuation or reflection coefficient) by allowing for complex $ABCD$ matrices. Hard-edged apertures and mirrors usually require numerical analysis [3,4]. Soft apertures can be treated analytically, and show broadly similar phenomena. Tovar and

Casperson [14,15] developed an extension of *ABCD* methods applicable to systems that exhibit both misalignment and space-dependent gain or loss. They form a compound (possibly complex) "*ABCDGH*" matrix describing propagation from input to output planes (e.g., a full round trip of a laser cavity) by ordered multiplication of elementary submatrices for components or sections of the system. In [15] a generalized Huygens' integral (HI) is derived, with the kernel having the usual *ABCD* dependence, but with extra terms dependent on G and H.

As in the *ABCD* case [16], it is always possible to symmetrize the kernel of this HI by introducing a "fictitious" inverse pair of complex lenses and placing the reference plane between them. The round-trip eigenmodes of this symmetric kernel then obey the nonconjugated orthogonality relation (1). Under this complex-lens transformation, the B element of the "physical" *ABCDGH* matrix is invariant, while the A and D elements become equal, with value $S = \cos\psi = (A + D)/2$, and the symmetrized HI is given by

$$E_{\rm out}(x) = e^{\Gamma}\sqrt{\frac{-ik}{2\pi B}}\int_{-\infty}^{\infty} dx_0 K_s(x, x_0) E_{\rm in}(x_0),$$
$$K_s(x, x_0) = e^{i\{k[S(x^2+x_0^2)-2xx_0]/2+\sigma(x+x_0)/B\}}. \tag{5}$$

Here $k = 2\pi/\lambda$ is the wave vector at the reference plane. σ is the "H" element of the transformed matrix, related to the original elements by $\sigma = [BG + (1 - A)H]/2$. Its presence thus requires that one or both of G and H be nonzero, i.e., some form of misalignment [14]. The prefactor e^{Γ} describes the effect of transversely uniform gain or loss elements, including, for example, a uniform gain set so as to place the laser exactly at threshold. Above threshold, saturation will reduce the gain and modify its transverse profile, but the evolution of perturbations incoherent with the lasing mode, such as noise, should still be reasonably well described by an HI of the form (5). It is easy to verify that (5) has a tilted Gaussian eigenfunction of the form (3), with

$$X + iY = \pm\frac{k\sin\psi}{B}, \tag{6a}$$

$$h = \frac{1}{X}\,{\rm Re}\left[\frac{\sigma}{B}\cot\left(\frac{\psi}{2}\right)\right], \tag{6b}$$

$$\mu = \left(\frac{X^2 + Y^2}{kX}\right){\rm Im}\left(\frac{\sigma}{1 - \cos\psi}\right). \tag{6c}$$

Because both B and S can be complex, and can lie in any quadrant, the sign in (6a) cannot be predetermined, but can always be chosen such that $X > 0$. Given these values for the mode parameters, using (4) we can readily evaluate the Petermann factor, which clearly has a superexponential dependence on the misalignment parameter σ. We can thus conclude that giant excess noise is generic in sufficiently misaligned laser cavities—provided that the K calculated at our symmetry plane is representative of that at any physical reference plane.

It has been shown that a cavity usually has as many Petermann factors as it has sections made disjoint by apertures [7,17], and that the spectral properties of a laser are determined by an "effective excess noise factor" [7], broadly the weighted mean of the Petermann factors over the amplifying plane(s) within the cavity. Our symmetrizing transformation is equivalent, in its effect on K, to an aperture offset by a distance l from the axis. A soft aperture that multiplies the mode U by a factor $e^{-\gamma(x-l)^2/2}$ scales K by

$$K_{ap} = \sqrt{X^2/\gamma^2 - 1}\,\exp\left[\frac{2\gamma^2 X(l - h)^2}{(X^2 - \gamma^2)}\right]. \tag{7}$$

Since $K_{\rm ap} \geq 1$, K *increases* in going from the "fictitious symmetry plane" to the adjacent physical plane. Where a noise source is in a cavity section disjoint from the reference (symmetry) plane, the change of K through the aperture exactly accounts for the change in the noise field in propagating from the source plane to the reference plane. We can thus conclude the following: at any and every reference plane of a misaligned laser cavity describable by a generalized *ABCD* matrix, the excess noise is enhanced by a factor that is exponential in the square of the misalignment parameter.

The above formulas establish K in terms of the profile, offset, and tilt of the *stationary* fundamental mode U of the laser described by (5). We can obtain further insight by allowing these parameters to evolve in time, during either the establishment of the mode from noise or the (linear) perturbation of the established mode by noise incoherent with it. It has previously been shown, for a continuous medium [18] and for compound *ABCD* systems [19], that an input perturbation may acquire excess amplification, and that the maximum value of this "weak wave excitation factor" is exactly equal to K. To extend this approach to misaligned systems, suppose that at $t = 0$ one injects a weak field with the form (3), but with arbitrary parameters $X_0 + iY_0$, h_0, and μ_0. Iteration of the HI (5) generates a mapping of these parameters, which evolve asymptotically to their stable fixed-point values given by (6). The ratio of the asymptotic and initial amplitudes largely determines the *transient gain,* defined as the fractional amplification of the perturbation's energy as it evolves towards the stable fundamental mode. Because n iterations of the single-trip HI are equivalent to a single HI with the kernel given by the nth power of the single-trip *ABCDGH* matrix, $E_{\rm out}^{(n)}$ can be exactly evaluated and the limit $n \to \infty$ taken. We optimize with respect to the initial parameters, and find that the maximum transient gain occurs for $h_0 = h$, i.e., for a perturbation with the same offset as the mode, but with $\mu_0 = -\mu$ (initial tilt exactly opposite to that of the mode). The optimum beam shape has $X_0 = X$ and $Y_0 = -Y$, in accord with previous analysis [18–20]. The associated maximum transient gain is then *exactly* $K = K^{\rm al}K^{\rm mis}$.

Following the dynamical evolution of the beam parameters we find that an initial "weak wave" with these optimal parameters drifts towards low offsets, thus accumulating significant differential gain, before stabilizing at the (finite) equilibrium offset. We can thus interpret the giant excess noise that can arise on misalignment as due to excess amplification of spontaneous emission (or other) noise as the noise field evolves towards the stable mode, consistent with, but generalizing, previous work [4,18–20]. This dynamical interpretation of excess noise seems more physical than one based on noise correlation between multiple modes. (How many modes would be needed to describe a K of 10^{10}, such as that found below?) It also leads, in its relation to transient gain and optimal perturbations, to a natural link with other fields such as fluid turbulence [11].

Because of the superexponential dependence of K^{mis} on the misalignment parameter σ, one could expect noise to begin to dominate the laser output as misalignment is increased, leading to a collapse of coherence. Such an effect is known to occur when the modulation frequency of an actively mode-locked laser is sufficiently detuned from the round-trip time. Kärtner *et al.* [9] have already identified this loss of coherence with transient gain associated with nonorthogonality of the eigenpulses, and Geddes *et al.* [10] showed that this transient gain is identical to the Petermann factor, generalizing K to the time domain. We now see that this identity, and thus giant excess noise, is a completely general effect of misalignment, whether in the time or the space domain.

To illustrate and confirm our general analysis we now calculate K for a class of Fabry-Perot cavity with a mirror of Gaussian reflectivity profile, devised and analyzed by Doumont *et al.* [5], and later used by Longhi to illustrate the effect of mirror tilt on K [13]. In fact, we consider a broadly equivalent system, shown in "unfolded" form in Fig. 1. It has two plane mirrors, one-way propagation between which is described by a real "$abcd$" matrix, augmented by a Gaussian aperture in front of one mirror. The key difference from [5] is that we allow this aperture to

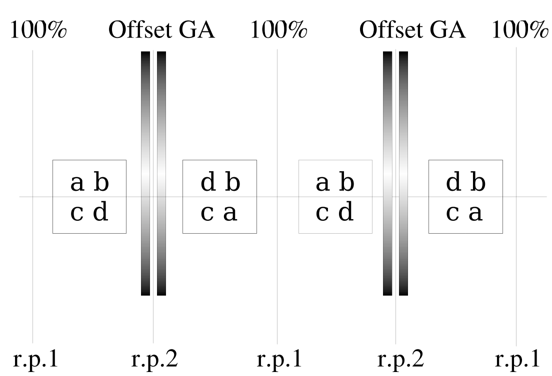

FIG. 1. Unfolded Fabry-Perot cavity (see text). The thin vertical lines represent plane mirrors, while the half-trip "$abcd$" and "$dbca$" matrices are real. An offset Gaussian aperture in front of one mirror replaces the Gaussian reflectivity mirror in the scheme of [5].

be *offset* by a distance l from the axis. It multiplies the incident field by a transmission factor $\exp[-\frac{\gamma}{2}(x-l)^2]$ on each pass, where $\gamma = 1/w_{ga}^2$ in the notation of Doumont *et al.* The round-trip HI for this cavity has a symmetric kernel at both the aperture mirror and the other mirror (the reference plane in [5,13]), allowing direct use of the theory developed above.

At the aperture-mirror plane (r.p.2 in Fig. 1) $B = B_2 = 2ab$, and by inspection the round-trip HI is

$$E_{\mathrm{out}} = e^{\Gamma}\sqrt{\frac{-ik}{2\pi B}}\int_{-\infty}^{\infty} dx_0 K_2(x, x_0) E_{\mathrm{in}}(x_0),$$
$$K_2(x, x_0) = e^{-\gamma(x-l)^2/2} e^{ik(mx^2+mx_0^2-2xx_0)/2B} e^{-\gamma(x_0-l)^2/2}. \tag{8}$$

We use the same cavity parameters as Doumont *et al.*, namely $m = ad + bc$ and $\delta = \gamma L/k$, where $L = 2ab$ is an effective cavity length. Clearly (8) can be put into the form (5), with $S = m + i\delta$ and $\sigma_2 = -ikl\delta$. Adopting $R = \mathrm{Re}(\sin\psi)$ as a convenient dimensionless parameter, $X_2 = |kR/2ab|$, and the Petermann factor at the aperture mirror is given by

$$K_2 = \frac{\sqrt{(R^2+m^2)^2 - m^2}}{|R|\sqrt{(R^2+m^2)}} \exp\left(\frac{\kappa_2 l^2}{\lambda L}\right), \tag{9a}$$

$$\kappa_2 = \frac{4\pi\delta^2(m-1)^2(R^2+m^2)(R^2+m^2+m)}{|R|(R^2+m^2-m)^3}. \tag{9b}$$

The HI at the other mirror (r.p.1 in Fig. 1) can be obtained by transformation of (8). It also has the symmetric form (5), but with $\sigma_1 = -ikl\delta/a$. Then

$$K_1 = \frac{\sqrt{(R^2+m^2)^2 - m^2}}{|R|} \exp\left(\frac{\kappa_1 l^2}{\lambda L}\right), \tag{10a}$$

$$\kappa_1 = \frac{8\pi\delta^2(m-1)^2(R^2+m^2)^2}{|R|(R^2+m^2-m)^3}. \tag{10b}$$

It can be verified that K_1 and K_2 are related by the scaling given in (7), which applies equally to real and fictitious apertures.

In Fig. 2 we plot K^{al} and K^{mis} at both reference planes, for fixed $\delta = 0.02$, against magnification M [defined as $M = m$ for $m \le 1$, $M = m + \sqrt{(m^2-1)}$ for $m > 1$]. K_1^{al} is exactly as in [5], but note that $K_2^{\mathrm{al}} \neq K_1^{\mathrm{al}}$. The excess noise is dominated by misalignment, however, peaking at $K = K^{\mathrm{al}}K^{\mathrm{mis}} > 10^{10}$ for m just above unity, i.e., for a cavity that is very slightly unstable in the absence of an aperture. These huge values arise when the aperture is offset by its width w_{ga}: they, of course, scale dramatically with offset.

At both planes the power loss is $1 - |e^{\nu}|^2$ where

$$\nu = \mp\frac{i\psi}{2} - \gamma l^2\left(\frac{m-1}{m-1+i\delta}\right). \tag{11}$$

The loss penalty of misaligning the aperture is evident, but note that it vanishes for $m = 1$, as expected. The huge excess noise in Fig. 2 occurs for misalignment power

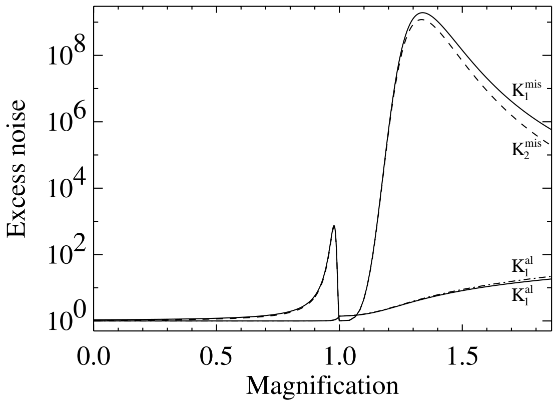

FIG. 2. Excess noise factors for the cavity of Fig. 1, plotted against magnification M for $\delta = 2ab/kw_{ga}^2$ (same parameters as in Fig. 1 of [5]). Upper curves K^{mis} show giant enhancement of excess noise for aperture offset by $l \approx w_{ga}$. Lower curves K^{al} are for $l = 0$, corresponding to [5]. In each case K_2 is evaluated at the aperture-mirror plane (Fig. 1), and K_1, which is somewhat larger, at the other mirror (the reference plane of [5]).

loss only about 4 times the loss of the aligned cavity, suggesting good prospects for experimental observation, and perhaps even exploitation, of giant Petermann factors.

For example, optically pumped vertical-external-cavity surface-emitting lasers (VECSELs) naturally have an apertured gain (which is broadly equivalent to a loss aperture [3]), and furthermore can easily be driven many times above threshold [21]. Our analysis suggests that operation of a VECSEL in a slightly unstable configuration ($m > 1$) should enable observation of a dramatic increase in noise as the pumping beam is progressively offset from the cavity axis. Perhaps more usefully, our dynamical analysis suggests that, below threshold, such a laser should act as a high-gain amplifier for a suitably injected signal beam. In contrast to a single-mode laser, such an amplifier would *not* exhibit excess noise, as is well known [18]. This is also evident from our transient gain analysis, which shows that the excess amplification depends on the configuration of the perturbation, but not on whether it is signal or noise.

In summary, we have shown that breaking the transverse symmetry of a laser cavity through misalignment can easily lead to giant values of transient gain and excess noise, for which we have derived analytic formulas. We calculated $K > 10^{10}$ for moderate misalignment of a typical cavity. Physical quantities of such magnitude are very unusual in the absence of singularities. We expect similarly dramatic enhancements of transient gain in other physical systems that exhibit nonorthogonal modes.

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[1] K. Petermann, IEEE J. Quantum Electron. **QE-15**, 566 (1979).
[2] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).
[3] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986).
[4] G. H. C. New, J. Mod. Opt. **42**, 799 (1995).
[5] J. Doumont, P. L. Mussche, and A. E. Siegman, IEEE J. Quantum Electron. **QE-25**, 1960 (1989).
[6] M. P. van Exter, N. J. van Druten, A. M. van der Lee, S. M. Dutra, G. Nienhuis, and J. P. Woerdman, Phys. Rev. A **63**, 043801 (2001).
[7] K. Joosten and G. Nienhuis, Phys. Rev. A **58**, 4937 (1998).
[8] A. M. van der Lee, M. P. van Exter, A. L. Mieremet, N. J. van Druten, and J. P. Woerdman, Phys. Rev. Lett. **81**, 5121 (1998).
[9] F. X. Kärtner, D. M. Zumbühl, and N. Matuschek, Phys. Rev. Lett. **82**, 4428 (1999).
[10] J. B. Geddes, W. J. Firth, and K. Black, SIAM J. Appl. Dyn. Sys. **2**, 647 (2003).
[11] D. Biau and A. Bottaro, Phys. Fluids **16**, 3515 (2004).
[12] L. N. Trefethen, SIAM Rev. **39**, 383 (1997).
[13] S. Longhi, Opt. Lett. **25**, 811 (2000).
[14] A. A. Tovar and L. W. Casperson, J. Opt. Soc. Am. A **12**, 1522 (1995).
[15] A. A. Tovar and L. W. Casperson, J. Opt. Soc. Am. A **13**, 2239 (1996).
[16] A. M. Dunlop, W. J. Firth, and E. M. Wright, Opt. Commun. **138**, 211 (1997).
[17] A. M. Dunlop, W. J. Firth, and E. M. Wright, in *Proceedings of the Conference on Lasers and Electro-Optics Europe* (IEEE, Piscataway, NJ, 1998), p. 82.
[18] H. A. Haus and S. Kawakami, IEEE J. Quantum Electron. **QE-21**, 63 (1985).
[19] A. E. Siegman, Phys. Rev. A **39**, 1253 (1989).
[20] A. E. Siegman, Phys. Rev. A **39**, 1264 (1989).
[21] H. Lindberg, M. Strassner, J. Bengtsson, and A. Larsson, IEEE Photonics Technol. Lett. **16**, 362 (2004).