Critical Behavior in Vacuum Gravitational Collapse in 4 + 1 Dimensions

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We show that the (4 + 1)-dimensional vacuum Einstein equations admit gravitational waves with radial symmetry. The dynamical degrees of freedom correspond to deformations of the three-sphere orthogonal to the (t, r) plane. Gravitational collapse of such waves is studied numerically and shown to exhibit discretely self-similar type II critical behavior at the threshold of black hole formation.

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Introduction and setup.—The gravitational collapse to a black hole is a subject of intensive studies in general relativity. One of the main goals of these studies is to prove the cosmic censorship conjecture, which says that a physically realistic generic gravitational collapse cannot result in a naked singularity. It would be most interesting to assert that this conjecture is true in vacuum; however, with current analytical techniques the problem seems tractable only in spherical symmetry, and in this case there is no vacuum collapse because of Birkhoff's theorem. Thus, in order to generate spherically symmetric dynamics one has to couple matter fields. A simple choice, which has led to important insights, is a real massless scalar field. For this matter model Christodoulou showed that for small initial data the fields disperse to infinity [1], while for large data black holes are formed [2]. The transition between these two outcomes of evolution was explored numerically by Choptuik [3], leading to the discovery of critical phenomena at the threshold of black hole formation. Similar phenomena were later observed in many other matter models in spherical symmetry (see [4] for a comprehensive review) but, because of numerical difficulties, only once in vacuum for axially symmetric gravitational waves [5].

The aim of this Letter is to show that—at the price of going to (4 + 1) dimensions—one can evade Birkhoff's theorem and have gravitational collapse of pure gravitational waves in radial symmetry. The idea is very simple and is based on the fact that the geometry of the three-sphere S^3 has the property that one can break the isotropy but still have a homogeneous space. This happens as follows. The group of rotations acting on S^3 in Euclidean space has a subgroup G_3 acting simply transitively on the three-sphere. This subgroup is isomorphic to the universal covering group of the connected component of the rotation group in three dimensions. The action of G_3 is generated by the simultaneous rotations in the (x-y, z-w) planes, the (x-z, y-w) planes and (x-w, z-y) planes [this action on S^3 defines also the Bianchi IX homogeneous cosmological model in the (3 + 1)-dimensional general relativity]. Thus, in (4 + 1) dimensions it is consistent to consider spacetimes with the metric of the form

$$ds^2 = -U(t,r)dt^2 + V(t,r)dr^2 + \sum_{k=1}^{3} L_k^2(t,r)\sigma_k^2, \quad (1)$$

where σ_k are three one-forms invariant under G_3 satisfying the commutation relations $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. In terms of Euler angles ($0 \le \theta \le \pi$, $0 \le \phi, \psi \le 2\pi$)

$$\sigma_1 + i\sigma_2 = e^{i\psi}(\cos\theta d\phi + i d\theta),$$
$$\sigma_3 = d\psi - \sin\theta d\phi. \quad (2)$$

The metric functions $L_k(t, r)$ are the three principal curvature radii of the squashed three-sphere. The case when all three L_k are equal corresponds to the standard spherically symmetric ansatz for which the Birkhoff theorem applies and the only solutions are Minkowski and Schwarzschild. However, if L_k are different we will obtain nontrivial vacuum solutions with gravitational radiation. In this Letter we restrict ourselves to a special case of the ansatz (1) in which $L_1 = L_2$. Using the coordinate freedom in the two-space orthogonal to the group orbit of G_3, we choose the volume radial coordinate $r = (\mathrm{vol}(S^3)/2\pi^2)^{1/3}$, and write the metric as

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \tfrac{1}{4}r^2[e^{2B}(\sigma_1^2 + \sigma_2^2) + e^{-4B}\sigma_3^2], \quad (3)$$

where A, δ, and B are functions of t and r. Note that for this ansatz the three-sphere has a residual isotropy of the twisted product $S^2 \times S^1$ (as in the Taub universe).

Substituting the ansatz (3) into the vacuum Einstein equations, we get the equations of motion for the functions $A(t, r)$, $\delta(t, r)$ and $B(t, r)$. In the following we use overdots and primes to denote ∂_t and ∂_r, respectively. The Hamiltonian and momentum constraints are

$$A' = -\frac{2A}{r} + \frac{1}{3r}(8e^{-2B} - 2e^{-8B}) - 2r(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2), \tag{4}$$

$$\dot{A} = -4rA\dot{B}B'. \tag{5}$$

The evolution equation for B has a form of the quasilinear wave equation

$$(e^{\delta}A^{-1}r^3\dot{B})^{\cdot} - (e^{-\delta}Ar^3B')' + \tfrac{4}{3}e^{-\delta}r(e^{-2B} - e^{-8B}) = 0. \tag{6}$$

In addition, we have a slicing condition for δ

$$\delta' = -2r(e^{2\delta}A^{-2}\dot{B}^2 + B'^2). \tag{7}$$

It is clear from the above equations that the only dynamical degree of freedom is the field B which plays a role similar to the spherically symmetric scalar field in $(3+1)$ dimensions. If $B = 0$, it is easy to verify (Birkhoff's theorem) that the only solution is Schwarzschild $\delta_0 = 0$, $A_0 = 1 - r_h^2/r^2$ (or Minkowski if $r_h = 0$). As we shall see below these two static solutions play the role of attractors. Note that Eqs. (4)–(7) are scale invariant, which excludes the existence of regular asymptotically flat static solutions.

It is convenient to introduce the mass function $m(t, r)$ defined by $A = 1 - m(t, r)/r^2$. Then, the Hamiltonian constraint (4) takes a simple form

$$m' = 2r^3(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2) + \tfrac{2}{3}r(3 + e^{-8B} - 4e^{-2B}). \tag{8}$$

Note that the right-hand side of Eq. (8) is manifestly positive so $m(t, r)$ is monotone increasing with r. For asymptotically flat spacetimes $m_\infty = \lim_{r\to\infty} m(t, r)$ exists and is constant in time. The total mass is given by $M = (3\pi/8G)m_\infty$.

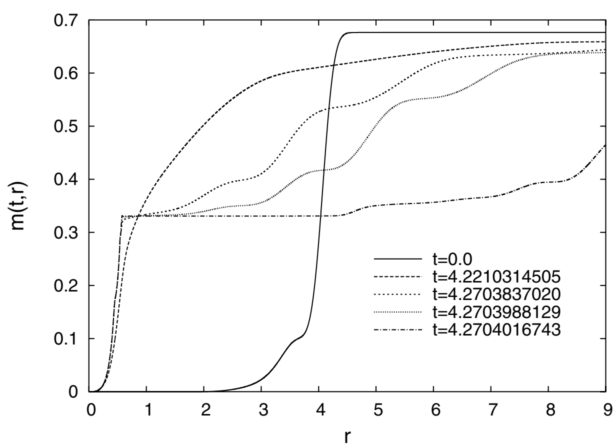

FIG. 1. Formation of a black hole for highly supercritical initial data. We plot the mass function $m(t, r)$ at the initial time and at four late times. The total mass (measured in units $3\pi/8G$) is 0.67 (see the plateau of the initial profile). During the evolution the function $m(t, r)$ develops a second inner plateau that indicates formation of the Schwarzschild black hole with mass $M_{\mathrm{BH}} = 0.33$.

We consider the initial value problem for the above equations. To ensure regularity at the center, we impose the boundary conditions

$$B(t, r) \sim b(t)r^2, \qquad 1 - A(t, r) \sim a(t)r^4. \tag{9}$$

We normalize time by the condition $\delta(t, 0) = 0$, which means that t is the proper time at the center.

Numerical results.—We have solved the above equations using the free evolution scheme in which $A(t, r)$ is updated using the momentum constraint (5). The Hamiltonian constraint (4) was solved at $t = 0$ and then monitored only to check the accuracy of the code. The wave Eq. (6) was rewritten as the pair of two first order equations for B and an auxiliary variable $P = e^{\delta}\dot{B}/A$. Integration in time was done by a modified predictor-corrector McCormack method on a uniform spatial grid. The ordinary differential Eq. (7) was solved with the fourth order Runge-Kutta method. The whole procedure has an accuracy of second order in time and fourth order in space.

The numerical results presented below were produced for the initial data of the form of an "ingoing" generalized Gaussian

$$B(0, r) = p\left(\frac{r}{r_0}\right)^4 e^{-(r-r_0)^4/s^4}, \quad P(0, r) = rB'(0, r)/r_0, \tag{10}$$

where the parameter p was varied and the parameters r_0 and s were fixed. To check the universality of the critical behavior, we have verified that the results are independent of the specific choice of initial data.

We have found that for all families of initial data the same picture, similar to the massless scalar field collapse, emerges. For small values of the control parameter p the fields disperse, leaving behind the flat spacetime. For large initial data a Schwarzschild black hole is formed, as shown in Figs. 1 and 2.

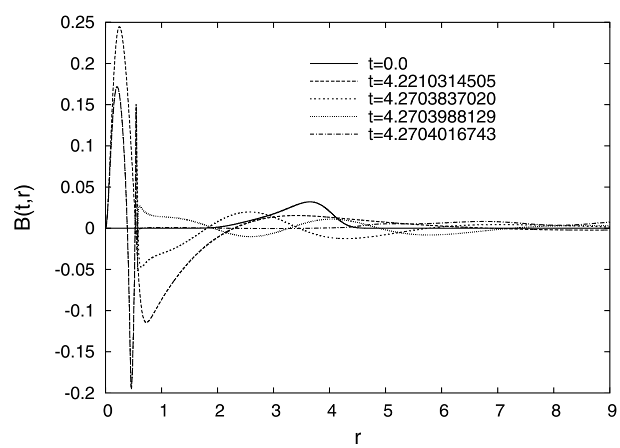

FIG. 2. The plot of $B(t, r)$ for the same data as in Fig. 1. Outside the horizon developing at $r_h = 0.57$, B tends to zero.

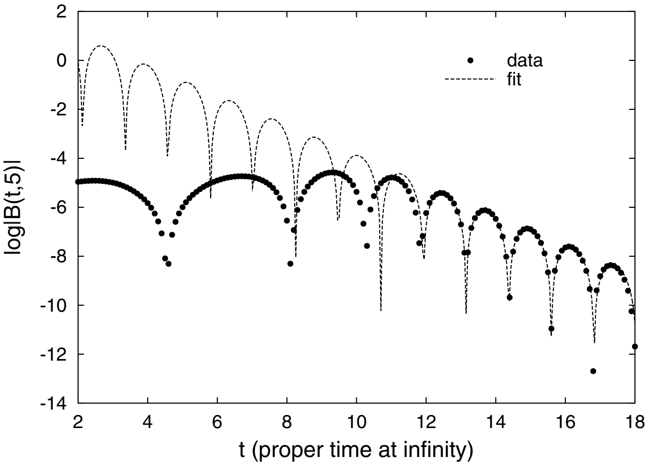

FIG. 3. Quasinormal ringing of the Schwarzschild black hole. We plot the time series $\ln|B(t, r_0)|$ at $r_0 = 5$ for the same data as in Fig. 2 but using the proper time at infinity. The horizon forms at $r_h = 0.575$; hence according to [6], the least damped quasinormal mode has eigenvalue $k = 2.62 - 0.62i$. The linear perturbation regime sets in at $t \sim 11$. The fit of an exponentially damped sinusoid to the data on the time interval $13 < t < 18$ gives $k = 2.56 - 0.61i$, which is in good agreement with the theoretical prediction.

The process of settling down to the Schwarzschild black hole can be described in more detail using linear perturbation theory. Linearizing Eqs. (4)–(7) around the Schwarzschild solution, we obtain the linear wave equation for the perturbation $\delta B(t, r)$,

$$\delta \ddot{B} - \frac{1}{r^3} A_0 (r^3 A_0 \delta B')' + \frac{8A_0}{r^2} \delta B = 0, \quad A_0 = 1 - \frac{1}{r^2}, \quad (11)$$

where we have used the scaling freedom to set the radius of the horizon $r_h = 1$. Introducing the tortoise coordinate $x = r + \frac{1}{2} \ln \frac{r-1}{r+1}$ and substituting $\delta B(t, r) = e^{-ikt} u(x)$ into (11) we get the Schrödinger equation on the real line $-\infty < x < \infty$,

$$-\frac{d^2 u}{dx^2} + V(r(x)) u = k^2 u, \quad (12)$$

where

$$V(r) = \frac{1}{4} \left(1 - \frac{1}{r^2}\right) \left(\frac{35}{r^2} + \frac{9}{r^4}\right). \quad (13)$$

Quasinormal modes [i.e., solutions of Eq. (12) satisfying the outgoing wave conditions $u \sim e^{\pm ikx}$ for $x \to \pm\infty$] for potentials of this type have been computed in [6] via the method of continued fractions. The potential (13) corresponds to the gravitational tensor perturbation with $l = 2$, and in this case the least damped mode (see Table III in [6]) has eigenvalue $k = 1.51 - 0.36i$ (in units r_h^{-1}). This mode is expected to dominate the intermediate asymptotics of local convergence to Schwarzschild. In Fig. 3 we show evidence confirming this expectation.

We turn now to the description of critical behavior at the threshold of black hole formation. In a now routine procedure for bistable systems, using bisection we have determined a critical parameter value p^* that separates black hole formation from dispersion. The behavior of near-critical solutions clearly indicates the existence of a discretely self-similar critical solution with the echoing period $\Delta \sim 0.47$. The profile of a near-critical solution and evidence for discrete self-similarity are shown in Figs. 4 and 5.

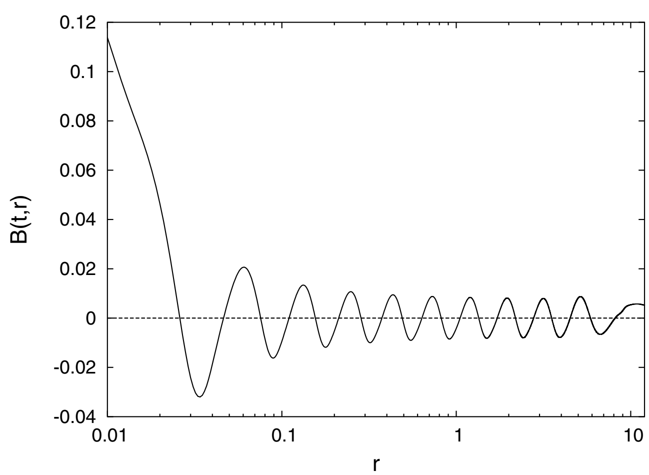

FIG. 4. The late time profile of B for a near-critical solution. We easily get many echoes even for a moderate fine-tuning because the product of the eigenvalue λ of the growing mode about the critical solution and the echoing period Δ is relatively small, $\lambda\Delta \approx 2.86$ (for comparison, this product is about 3 times greater for the massless scalar field critical collapse [3]).

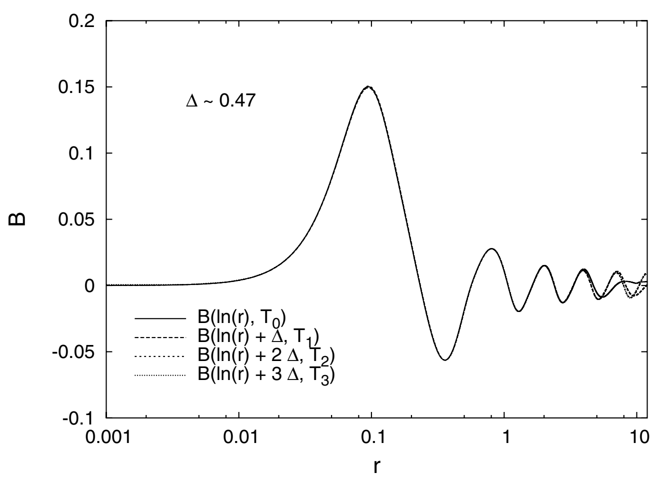

FIG. 5. Evidence for discrete self-similarity of the critical solution. For a near-critical evolution we plot the profile of B at some arbitrary central proper time T_0 and superimpose the next three echoes that subsequently develop. The times T_1, T_2, T_3 and the echoing period Δ were chosen to minimize $B(\ln(r) + n\Delta, T_n) - B(\ln(r), T_0)$.

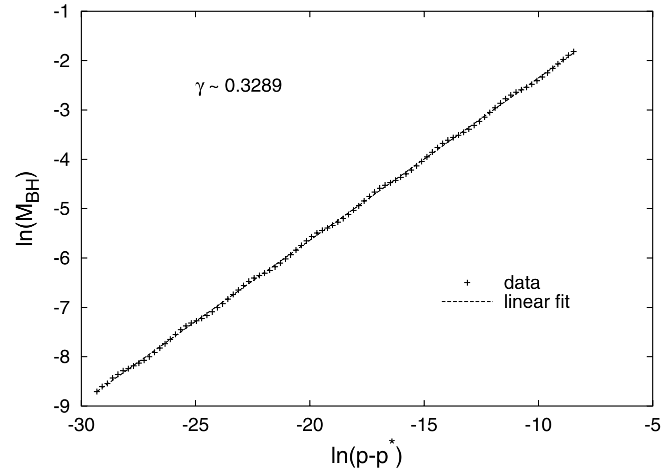

FIG. 6. Black hole mass scaling. For supercritical solutions the logarithm of black hole mass $M_{\rm BH}$ (measured in units $3\pi/8G$) is plotted versus the logarithmic distance to criticality. The slope of the linear fit yields $\gamma \approx 0.3289$. The wiggles, which are imprints of discrete self-similarity, are shown in more detail in Fig. 7.

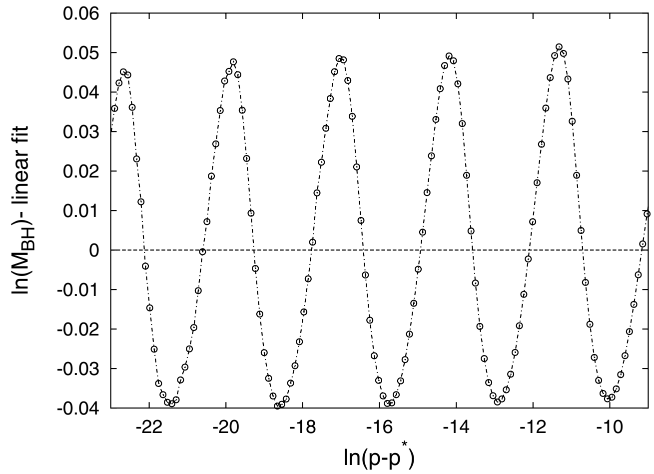

FIG. 7. Fine structure of black hole mass scaling. The linear fit from Fig. 6 is subtracted from the data. The period of the wiggles agrees to two decimal places with the theoretical prediction $\Delta/\gamma = 1.43$.

As expected, the mass of the black hole $M_{\rm BH}(p)$ changes continuously with p and tends to zero for $p \to p^*$ according to the power law

$$M_{\rm BH} \sim (p^* - p)^{\gamma}, \tag{14}$$

where the scaling exponent $\gamma \sim 0.3289$ is universal (i.e., independent of initial data). This is shown in Figs. 6 and 7. Note that, in four space dimensions, mass has the dimension of length^2; hence $\gamma = 2/\lambda$, where λ is an eigenvalue of the growing mode of the critical solution.

To summarize, we have studied gravitational collapse of pure gravitational waves in (4 + 1) dimensions and have found strong evidence for the type II discretely self-similar critical behavior at the threshold of black hole formation. As far as we know, besides the notable paper [5], this is the only example of critical behavior without matter.

Concluding, let us mention some natural extensions of the study presented here. One interesting possibility is to investigate the general ansatz (1) with two dynamical degrees of freedom—the studies in this direction are in progress and will be reported elsewhere. It is also natural to look for similar models in higher dimensions. If one insists that the subgroup of the orthogonal group acts simply transitively, the only possibility is the one discussed in our Letter. However, if one allows multiply transitive subgroups of the orthogonal groups, the results given in Besse [7] show that there are models similar to the one considered here on any odd-dimensional sphere.

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