New Einstein-Sasaki Spaces in Five and Higher Dimensions

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We obtain infinite classes of new Einstein-Sasaki metrics on complete and nonsingular manifolds. They arise, after Euclideanization, from BPS limits of the rotating Kerr–de Sitter black hole metrics. The new Einstein-Sasaki spaces $L^{p,q,r}$ in five dimensions have cohomogeneity 2 and $U(1) \times U(1) \times U(1)$ isometry group. They are topologically $S^2 \times S^3$. Their AdS/CFT duals describe quiver theories on the four-dimensional boundary of AdS_5. We also obtain new Einstein-Sasaki spaces of cohomogeneity n in all odd dimensions $D = 2n + 1 \geq 5$, with $U(1)^{n+1}$ isometry.

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The AdS/CFT correspondence [1] relates type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{K}_5$ backgrounds to conformal field theories on the four-dimensional boundary of the AdS_5, where K_5 is a complete five-dimensional Einstein space of positive Ricci curvature. To maintain supersymmetry, it is necessary that K_5 admit a Killing spinor; i.e., that it be an Einstein-Sasaki (ES) space. Thus, the construction of such ES spaces provides an important testing ground for the AdS/CFT correspondence.

The most studied case is when K_5 is taken to be the 5-sphere, which admits the maximal number, 4, of Killing spinors, and it has an isometry of $SO(6)$. The corresponding boundary theory is an $\mathcal{N} = 4$ supersymmetric superconformal field theory. Another extensively studied case is when K_5 is $T^{1,1}$, which is an homogeneous ES space with $SU(2)^2 \times U(1)$ isometry. Recently, an infinite class of five-dimensional ES spaces was obtained [2]. These spaces, denoted by $Y^{p,q}$, are characterized by two coprime positive integers p and q with $q < p$. In the construction in [2], a local family of ES metrics with a nontrivial continuous parameter was first obtained, and then it was shown that if the parameter takes rational values p/q in the appropriate range, the metrics extend smoothly onto the complete and nonsingular manifolds $Y^{p,q}$. The spaces have $SU(2) \times U(1) \times U(1)$ isometry. These manifolds were shown to be dual to an infinite family of superconformal quiver gauge theories [3].

It is therefore of great interest to construct new ES spaces by further reducing the isometry. It was shown [4] that the ES spaces $Y^{p,q}$ could be obtained in a straightforward manner by taking a certain limit of the Euclideanized five-dimensional Kerr–de Sitter black hole metrics found in [5]. Specifically, after Euclideanization the two rotation parameters a and b were set equal and were allowed to approach the limiting value that corresponds, in the Lorentzian regime, to having rotation at the speed of light at infinity.

In this Letter, we construct a vastly greater number of Einstein-Sasaki spaces, in which a similar limit is taken but without requiring the rotation parameters to be equal. By this means, we first obtain a family of five-dimensional local ES metrics with two nontrivial continuous parameters. We then show that if these are appropriately restricted to be rational, the metrics extend smoothly onto complete and nonsingular manifolds, which we denote by $L^{p,q,r}$, where p, q, and r are coprime positive integers with $0 < p \leq q$, with $0 < r < p + q$, and with p and q each coprime to r and to $s = p + q - r$. The metrics have $U(1) \times U(1) \times U(1)$ isometry in general, enlarging to $SU(2) \times U(1) \times U(1)$ in the special case $p + q = 2r$, which reduces to the previously obtained spaces $Y^{p,q} = L^{p-q,p+q,p}$. The new ES spaces $L^{p,q,r}$ provide a greatly enlarged class of backgrounds for testing the AdS/CFT correspondence, by matching them to the boundary dual superconformal quiver gauge theories.

The local ES metrics that we construct are obtained from the rotating AdS black hole metrics in $D = 5$ dimensions [5] and in $D > 5$ [6,7]. Our principal focus is on the Euclidean-signature case with positive Ricci curvature, but it is helpful to think first of the metrics in the Lorentzian regime, with negative cosmological constant $\lambda = -g^2$. It was shown in [8] that the energy and angular momenta of the $D = 2n + 1$ dimensional Kerr–AdS black holes are given by

$$E = \frac{m\mathcal{A}_{D-2}}{4\pi(\prod_j \Xi_j)}\left(\sum_{i=1}^{n}\frac{1}{\Xi_i} - \frac{1}{2}\right), \qquad J_i = \frac{m a_i \mathcal{A}_{D-2}}{4\pi \Xi_i (\prod_j \Xi_j)}, \quad (1)$$

where $\mathcal{A}_{D-2}$ is the volume of the unit $(D-2)$ sphere, $\Xi_i = 1 - g^2 a_i^2$, and a_i are the n independent rotation parameters. As discussed in [9], the BPS limit can be found by studying the eigenvalues of the Bogomol'nyi matrix arising in the AdS superalgebra from the anticommutator of the supercharges. In $D = 5$, these eigenvalues are then proportional to $E \pm gJ_1 \pm gJ_2$. The BPS limit is achieved when one or more of the eigenvalues vanish. For just one zero eigenvalue, the four cases are equivalent under reversals of the angular velocities, so we may without loss of generality consider $E - gJ_1 - gJ_2 = 0$. From (1), we see that this is achieved by taking a limit in which ga_1 and ga_2 tend to unity, namely, by setting $ga_1 = 1 - \frac{1}{2}\epsilon\alpha$, $ga_2 = 1 - \frac{1}{2}\epsilon\beta$, rescaling m according to $m = m_0\epsilon^3$, and sending ϵ to zero. As we shall see, the metric remains nontrivial in this limit. An equivalent discussion in the Euclidean regime leads to the conclusion that in the corresponding limit, one obtains five-dimensional Einstein metrics admitting a Killing spinor. (The above scaling limit in the Lorentzian regime, for the special case $\alpha = \beta$, was studied recently in [10].)

To present our new ES metrics, we start with the five-dimensional rotating AdS black hole solutions, and Euclideanize by making the analytic continuations $t \to i\tau/\sqrt{\lambda}$, $\ell \to i\sqrt{\lambda}$, $a \to ia$, $b \to ib$ in the metric (5.22) of [5]. Next, we implement the "BPS scaling limit," by setting

$$a = \lambda^{-1/2}\left(1 - \frac{1}{2}\alpha\epsilon\right), \qquad b = \lambda^{-1/2}\left(1 - \frac{1}{2}\beta\epsilon\right),$$
$$r^2 = \lambda^{-1}(1 - x\epsilon), \qquad M = \frac{1}{2}\lambda^{-1}\mu\epsilon^3 \tag{2}$$

and then sending $\epsilon \to 0$. The metric becomes

$$\lambda ds_5^2 = (d\tau + \sigma)^2 + ds_4^2, \tag{3}$$

where

$$ds_4^2 = \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2$$
$$+ \frac{\Delta_\theta \sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2,$$
$$\sigma = \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi, \tag{4}$$
$$\Delta_x = x(\alpha - x)(\beta - x) - \mu, \qquad \rho^2 = \Delta_\theta - x,$$
$$\Delta_\theta = \alpha\cos^2\theta + \beta\sin^2\theta$$

It is easy to check that the four-dimensional metric in (4) is Einstein. The parameter μ is trivial, and can be set to any nonzero constant, say $\mu = 1$, by rescaling α, β, and x. (The round S^5 arises when $\mu = 0$.) The metrics depend on two nontrivial parameters, which we can take to be α and β at fixed μ. It is sometimes convenient to retain μ, allowing it to be determined as the product of the three roots x_i of Δ_x.

It is clear that the $D = 5$ metric can be viewed as a $U(1)$ bundle over a four-dimensional Einstein-Kähler metric, with Kähler 2-form given by $J = \frac{1}{2}d\sigma$. Thus, the five-dimensional metric is ES (with $R_{\mu\nu} = 4\lambda g_{\mu\nu}$).

Having obtained the local form of the $D = 5$ ES metrics, we can now turn to an analysis of the global structure. The metrics are, in general, of cohomogeneity 2, with toric principal orbits $U(1) \times U(1) \times U(1)$. The orbits degenerate at $\theta = 0$ and $\theta = \frac{1}{2}\pi$, and at the roots of the cubic function Δ_x appearing in (4). In order to obtain metrics on complete nonsingular manifolds, one must impose appropriate conditions to ensure that the collapsing orbits extend smoothly, without conical singularities, onto the degenerate surfaces. If this is achieved, one can obtain a metric on a nonsingular manifold, with $0 \le \theta \le \frac{1}{2}\pi$ and $x_1 \le x \le x_2$, where x_1 and x_2 are two adjacent real roots of Δ_x. In fact, since Δ_x is negative at large negative x and positive at large positive x, and since we must also have $\Delta_x > 0$ in the interval $x_1 < x < x_2$, it follows that x_1 and x_2 must be the smallest two roots of Δ_x.

The easiest way to analyze the behavior at each collapsing orbit is to examine the associated Killing vector ℓ whose length vanishes at the degeneration surface. By normalizing the Killing vector so that its "surface gravity" κ is equal to unity, one obtains a translation generator $\partial/\partial\chi$ where χ is a local coordinate near the degeneration surface, and the metric extends smoothly onto the surface if χ has period 2π. The "surface gravity" is

$$\kappa^2 = \frac{g^{\mu\nu}(\partial_\mu \ell^2)(\partial_\nu \ell^2)}{4\ell^2} \tag{5}$$

in the limit that the degeneration surface is reached.

The normalized Killing vectors that vanish at the degeneration surfaces $\theta = 0$ and $\theta = \frac{1}{2}\pi$ are simply given by $\partial/\partial\phi$ and $\partial/\partial\psi$, respectively. At the degeneration surfaces $x = x_1$ and $x = x_2$, we find that the associated normalized Killing vectors ℓ_1 and ℓ_2 are given by

$$\ell_i = c_i\frac{\partial}{\partial\tau} + a_i\frac{\partial}{\partial\phi} + b_i\frac{\partial}{\partial\psi}, \tag{6}$$

where the constants c_i, a_i, and b_i are given by

$$a_i = \frac{\alpha c_i}{x_i - \alpha}, \qquad b_i = \frac{\beta c_i}{x_i - \beta},$$
$$c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)x_i - \alpha\beta - 3x_i^2}. \tag{7}$$

Since we have a total of four Killing vectors $\partial/\partial\phi$, $\partial/\partial\psi$, ℓ_1, and ℓ_2 that span a three-dimensional space, there must exist a linear relation among them. Since they all generate translations with a 2π period repeat, it follows that unless the coefficients in the linear relation are rationally related, then, by taking integer combinations of translations around the 2π circles, one could generate a translation implying an identification of arbitrarily nearby points in the manifold. Thus one has the requirement for obtaining a nonsingular manifold that the linear relation between the four Killing vectors must be expressible as

$$p\ell_1 + q\ell_2 + r\frac{\partial}{\partial\phi} + s\frac{\partial}{\partial\psi} = 0 \tag{8}$$

for *integer* coefficients (p, q, r, s), which may be assumed to be coprime. All subsets of three of the four integers must be coprime too, since if any three had a common divisor k, then dividing (8) by k would show that the direction associated with the Killing vector whose coefficient was not divisible by k would be identified with period $2\pi/k$, thus leading to a conical singularity. Furthermore, p and q must each be coprime to each of r and s, since otherwise at the surfaces where $\theta = 0$ or $\frac{1}{2}\pi$ *and* $x = x_1$ or $x = x_2$— at which one of $\partial/\partial\phi$ or $\partial/\partial/\psi$ *and* simultaneously one of ℓ_1 or ℓ_2 vanish—there would be conical singularities (see also [11,12]).

From (8) and (6), we have

$$pa_1 + qa_2 + r = 0, \qquad pb_1 + qb_2 + s = 0, \qquad pc_1 + qc_2 = 0. \tag{9}$$

It then follows that all ratios among the four quantities

$$a_1c_2 - a_2c_1, \qquad b_1c_2 - b_2c_1, \qquad c_1, \qquad c_2 \tag{10}$$

must be rational. Thus to obtain a metric that extends smoothly onto a complete and nonsingular manifold, we must choose the parameters in (4) so that the rationality of the ratios is achieved. In fact, it follows from (7) that

$$1 + a_i + b_i + 3c_i = 0, \tag{11}$$

for all roots x_i, and using this one can show that there are only two independent rationality conditions following from the requirements of rational ratios for the four quantities in (10). One can also see from (11) that

$$p + q - r - s = 0, \tag{12}$$

so the further requirement that all triples among the (p, q, r, s) also be coprime is automatically satisfied.

The upshot from the above discussion is that we can have complete and nonsingular five-dimensional ES spaces $L^{p,q,r}$, where

$$pc_1 + qc_2 = 0, \qquad pa_1 + qa_2 + r = 0. \tag{13}$$

These equations and (11) allow one to solve for α, β and the roots x_1 and x_2 for positive coprime integer triples (p, q, r). The requirements $0 \le x_1 \le x_2 \le x_3$ and $\alpha \ge x_2$, $\beta \ge x_2$ restrict the integers to the domains $0 < p \le q$ and $0 < r < p + q$. All such coprime triples with p and q each coprime to r and s yield complete and nonsingular ES spaces $L^{p,q,r}$, and so we get infinitely many new examples.

The volume of $L^{p,q,r}$ (with $\lambda = 1$) is given by

$$V = \frac{\pi^2(x_2 - x_1)(\alpha + \beta - x_1 - x_2)\Delta\tau}{2k\alpha\beta}, \tag{14}$$

where $\Delta\tau$ is the period of the coordinate τ, and $k = \gcd(p, q)$. Note that the (ϕ, ψ) torus is factored by a freely acting Z_k, along the diagonal. $\Delta\tau$ is given by the minimum repeat distance of $2\pi c_1$ and $2\pi c_2$, and so $\Delta\tau = 2\pi k|c_1|/q$. There is a quartic equation expressing V purely in terms of (p, q, r). Writing $V = \pi^3(p + q)^3W/(8pqrs)$, we find

$$\begin{aligned} 0 = {}& (1 - f^2)(1 - g^2)h_-^4 + 2h_-^2[2(2 - h_+)^2 - 3h_-^2]W \\ &+ [8h_+(2 - h_+)^2 - h_-^2(30 + 9h_+)]W^2 \\ &+ 8(2 - 9h_+)W^3 - 27W^4, \end{aligned} \tag{15}$$

where $f = (q - p)/(p + q)$, $g = (r - s)/(p + q)$, and $h_\pm = f^2 \pm g^2$. The central charge of the dual field theory is rational if W is rational, which is easily achieved.

If one sets $p + q = 2r$, i.e., $r = s$, implying α and β become equal, our ES metrics reduce to those in [2], and the conditions we have discussed for achieving complete nonsingular manifolds reduce to the conditions for the $Y^{p,q}$ obtained there, with $Y^{p,q} = L^{p-q,p+q,p}$. The quartic (15) then factorizes to quadratics with rational coefficients, giving the volumes found in [2].

Further special limits also arise. For example, if we take $p = q = r = 1$, the roots x_1 and x_2 coalesce, $\alpha = \beta$, and the metric becomes the homogeneous $T^{1,1}$ space, with the four-dimensional base space being $S^2 \times S^2$. In another limit, we can set $\mu = 0$ in (4) and obtain the round metric on S^5, with CP^2 as the base. (In fact, we obtain S^5/Z_q if $p = 0$.) Except in these special "regular" cases, the four-dimensional base spaces themselves are singular, even though the ES spaces $L^{p,q,r}$ are nonsingular. The ES space is called quasiregular if $\partial/\partial\tau$ has closed orbits, which happens if c_1 is rational. If c_1 is irrational, the orbits of $\partial/\partial\tau$ never close, and the ES space is called irregular.

Our construction generalizes straightforwardly to all odd higher dimensions $D = 2n + 1$. We take the rotating Kerr–de Sitter metrics obtained in [6,7], and impose the Bogomol'nyi conditions $E - g\Sigma_i J_i = 0$, where E and J_i are the energy and angular momenta that were calculated in [8], and given in (1). We find that a nontrivial BPS limit exists where $ga_i = 1 - \frac{1}{2}\alpha_i\epsilon$ and $m = m_0\epsilon^{n+1}$. After Euclideanization, we obtain $D = 2n + 1$ dimensional ES metrics ds^2, given by

$$\lambda ds^2 = (d\tau + \sigma)^2 + d\bar{s}^2, \tag{16}$$

with $R_{\mu\nu} = 2n\lambda g_{\mu\nu}$, where the $2n$-dimensional metric $d\bar{s}^2$ is Einstein-Kähler, with Kähler form $J = \frac{1}{2}d\sigma$, and

$$\begin{aligned} d\bar{s}^2 = {}& \frac{Ydx^2}{4xF} - \frac{x(1-F)}{Y}\Big(\sum_i \alpha_i^{-1}\mu_i^2 d\varphi_i\Big)^2 \\ &+ \sum_i (1 - \alpha_i^{-1}x)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \\ &+ \frac{x}{\sum_i \alpha_i^{-1}\mu_i^2}\Big(\sum_j \alpha_j^{-1}\mu_j d\mu_j\Big)^2 - \sigma^2, \end{aligned} \tag{17}$$

$$\sigma = \sum_i (1 - \alpha_i^{-1}x)\mu_i^2 d\varphi_i, \qquad Y = \sum_i \frac{\mu_i^2}{\alpha_i - x},$$
$$F = 1 - \frac{\mu}{x}\prod_i (\alpha_i - x)^{-1},$$

where $\Sigma_i \mu_i^2 = 1$.

The discussion of the global properties is completely analogous to the one we gave previously for the five-dimensional case. The n Killing vectors $\partial/\partial\varphi_i$ vanish at the degenerations of the $U(1)^{n+1}$ principal orbits at $\mu_i = 0$, and conical singularities are avoided if each coordinate φ_i has period 2π. The Killing vectors

$$\ell_i = c(i)\frac{\partial}{\partial\tau} + \sum_j a_j(i)\frac{\partial}{\partial\varphi_j} \tag{18}$$

vanish at the roots $x = x_i$ of $F(x)$, and have unit surface gravities there, where

$$a_j(i) = -\frac{c(i)\alpha_j}{\alpha_j - x_i}, \qquad c(i)^{-1} = \sum_j \frac{x_i}{\alpha_j - x_i} - 1. \tag{19}$$

The metrics extend smoothly onto complete and nonsingular manifolds if $p\ell_1 + q\ell_2 + \Sigma_j r_j \partial/\partial\varphi_j = 0$ for coprime integers (p, q, r_j), with p and q each coprime to each of the r_i. This implies the algebraic equations

$$pc(1) + qc(2) = 0, \qquad pa_j(1) + qa_j(2) + r_j = 0, \tag{20}$$

determining the roots x_1 and x_2, and the parameters α_j. The two roots of $F(x)$ must be chosen so that $F > 0$ when $x_1 < x < x_2$. With these conditions satisfied, we obtain infinitely many new complete and nonsingular ES spaces in all odd dimensions $D = 2n + 1$. Since it follows from (20) that $p + q = \Sigma_j r_j$, these ES spaces, which we denote by $L^{p,q,r_1,\ldots,r_{n-1}}$, are characterized by specifying $(n + 1)$ coprime integers, with p and q each coprime to each r_i, which must lie in an appropriate domain. The n torus of the φ_j coordinates is, in general, factored by a freely acting Z_k, where $k = \gcd(p, q)$. The volume (with $\lambda = 1$) is given by

$$V = \frac{|c(1)|}{q}\mathcal{A}_{2n+1}\left[\prod_i\left(1 - \frac{x_1}{\alpha_i}\right) - \prod_i\left(1 - \frac{x_2}{\alpha_i}\right)\right], \tag{21}$$

since $\Delta\tau$ is given by $2\pi k|c(1)|/q$, and $\mathcal{A}_{2n+1}$ is the volume of the unit $(2n + 1)$ sphere. In the special case that the rotations α_i are set equal, the metrics reduce to those obtained in [13].

Finally, we note that we also obtain new complete and nonsingular Einstein spaces in $D = 2n + 1$ that are not ES, by taking the Euclideanized Kerr–de Sitter metrics of [6,7] and applying the analogous criteria for nonsingularity at degenerate orbits that we have introduced in this Letter. Thus we Euclideanize the metrics given in Eq. (3.5) of [6] by sending $\tau \to -i\tau$, $a_i \to i\alpha_i$, take Killing vectors ℓ_i that vanish on two adjacent horizons, have unit surface gravities, obtained from the ℓ given in Eq. (4.7) of [6] by dividing by the surface gravity in Eq. (4.17), and then impose the rationality conditions following from $p\ell_1 + q\ell_2 + \Sigma_j r_j\partial/\partial\varphi_j = 0$. This gives infinitely many new examples of complete and nonsingular Einstein spaces, beyond those obtained in [6]. They are characterized by $(n + 2)$ coprime integers, and we denote them by $K^{p,q,r_1,\ldots,r_n}$.

Further details of these results will appear in [14].

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