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Nearly Smooth Granular Gases

Isaac Goldhirsch,[1,*] S. H. Noskowicz,[1,†] and O. Bar-Lev[2,‡]

[1]*Department of Fluid Mechanics and Heat Transfer, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel*
[2]*School of Mathematics, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel*

Hydrodynamic equations for nearly smooth granular gases are derived from the pertinent Boltzmann equation. The angular velocity distribution field needs to be included in the set of hydrodynamic fields. The angular velocity distribution is strongly non-Maxwellian for the homogeneous cooling state and *any* homogeneous steady state. In the case of steady wall-bounded shear flows the average spin (created at the boundaries) has a finite penetration length into the bulk.

In spite of many successes of models of granular gases in which tangential restitution is not accounted for [1], it is clear that friction is always present and that in numerous cases (cf., e.g., the recent study of the effects of friction on granular patterns [2] and friction induced hysteresis [3]) it is consequential. Furthermore, it is known that friction induces nonequipartition in the homogeneous cooling state; cf., e.g., [4–8]. It is therefore curious that only a small proportion of the literature is devoted to frictional granular hydrodynamics; cf., e.g., [9–11].

The study of gases whose constituents experience frictional interactions started (in 1894) in the realm of molecular gases [12,13]. Applications include granular celestial systems [14]. Previous kinetic theoretical studies of frictional granular hydrodynamics, e.g., [10,11], used the Enskog equations, in conjunction with an ansatz for the form of the distribution function, to produce constitutive relations. Their ansatz is based on Maxwellian distributions for both the velocity and angular velocity. It is *a priori* justified in the near-rough near-elastic case (strong rotation-translation coupling), where near equipartition is expected, but not in the near-smooth case. Unlike in the molecular case, the *model* commonly used for the description of granular gases is that of *smooth* particles (usually spheres). The goal here is to study the near-smooth case, as a perturbation around the smooth limit. Specifically, we consider a monodisperse system of spherical grains of mass $m = 1$, diameter d, and moment of inertia I [for homogeneous spheres, $I = \frac{2}{5}(\frac{d}{2})^2$] each. Denote the gyration ratio of a grain by $\tilde{I}$ (with $\tilde{I} \equiv \frac{4I}{d^2}$). The velocity of particle "i" is denoted by $\mathbf{v}_i$, and its angular velocity by ω_i. It is convenient to define a "spin variable," $\mathbf{s}_i \equiv \frac{d}{2}\omega_i$.

Consider a collision between sphere "1" and sphere "2." Let $\mathbf{k}$ be a unit vector pointing from the center of sphere 2 to the center of sphere 1. The relative velocity of sphere 1 with respect to sphere 2, at the point of contact, is $\mathbf{g}_{12} = \mathbf{v}_{12} + \mathbf{k} \times \mathbf{s}_{12}$, where $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{s}_{12} \equiv \mathbf{s}_1 + \mathbf{s}_2$. In the following, *precollisional entities are primed*. The collision model employed here is the same as in [6,11]. During a collision the normal component of the relative velocity changes according to $\mathbf{k} \cdot \mathbf{g}_{12} =$ $-e\mathbf{k} \cdot \mathbf{g}'_{12}$, where e is the coefficient of normal restitution. The change of the tangential component of $\mathbf{g}$ is modeled by $\mathbf{k} \times (\mathbf{k} \times \mathbf{g}_{12}) = -\beta(\gamma)\mathbf{k} \times (\mathbf{k} \times \mathbf{g}'_{12})$, where γ is the angle between $-\mathbf{k}$ and $\mathbf{g}'_{12}$: $\cos\gamma \equiv -\frac{\mathbf{k} \cdot \mathbf{g}'_{12}}{g'_{12}} = -\frac{\mathbf{k} \cdot \mathbf{v}'_{12}}{g'_{12}}$, where $g'_{12} \equiv \|\mathbf{g}'_{12}\|$ (hence $0 \leq \gamma \leq \frac{\pi}{2}$), and

$$\beta(\gamma) = \min\left\{\beta_0, -1 + \frac{1 + \tilde{I}}{\tilde{I}}(1 + e)\mu_f \cot\gamma\right\}, \quad (1)$$

where μ_f is the Coulomb friction coefficient, and β_0 is the "roughness coefficient" (which is not truly a constant in reality; cf., e.g., [15]; note also that the model used here is a simplification of reality; see, e.g., [16] and references therein). Equation (1) follows from the requirement that $\beta(\gamma)$ is continuous across the transition [at $\gamma = \gamma_0$, where from Eq. (1) $\cot\gamma_0 = \frac{\tilde{I}}{1+\tilde{I}} \frac{1+\beta_0}{\mu_f(1+e)}$] between dominance by (Coulomb) sliding for $\gamma > \gamma_0$, and "sticking" for $\gamma \leq \gamma_0$; see [6,11]. This model and the conservation laws imply

$$\mathbf{v}_i = \mathbf{v}'_i - \sigma_i \frac{1 + e}{2}(\mathbf{k} \cdot \mathbf{g}'_{12})\mathbf{k} + \sigma_i \frac{\tilde{I}[1 + \beta(\gamma)]}{2(1 + \tilde{I})}\mathbf{k} \times (\mathbf{k} \times \mathbf{g}'_{12}),$$

$$\mathbf{s}_i = \mathbf{s}'_i + \frac{1}{2}\frac{1 + \beta(\gamma)}{1 + \tilde{I}}\mathbf{k} \times \mathbf{g}'_{12}, \quad (2)$$

where $i = 1, 2$, $\sigma_1 = 1$, and $\sigma_2 = -1$. The Jacobian of this transformation is given by

$$J(\gamma) \equiv \frac{\partial(\mathbf{v}_1, \mathbf{v}_2, \mathbf{s}_1, \mathbf{s}_2)}{\partial(\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{s}'_1, \mathbf{s}'_2)} = \left\{\begin{array}{ll} e\beta_0^2 & \gamma < \gamma_0 \\ e|\beta(\gamma)| & \gamma > \gamma_0 \end{array}\right\}. \quad (3)$$

Let $f(\mathbf{v}_1, \mathbf{s}_1, \mathbf{r}, t)(\equiv f_1)$ denote the single particle distribution (of the velocity and spin) function at point $\mathbf{r}$ and time t. The Boltzmann equation satisfied by f_1 is

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \nabla f_1 = B(f, f, \mathbf{v}_1, \mathbf{s}_1)$$

$$\equiv d^2 \int_{\mathbf{k} \cdot \mathbf{v}_{12} > 0} d\mathbf{v}_2 d\mathbf{s}_2 d\mathbf{k}(\mathbf{k} \cdot \mathbf{v}_{12})$$

$$\times \left(\frac{1}{eJ(\gamma)}f'_1 f'_2 - f_1 f_2\right). \quad (4)$$

The basic premise of the Chapman-Enskog expansion is that the dependence of the distribution function on space

(i.e., $\mathbf{r}$) and time, t, can be replaced by a dependence on the "slow" fields, i.e., the densities of the (collisional) conserved entities. Therefore the number density, $n(\mathbf{r}, t) \equiv \int f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t)d\mathbf{v}d\mathbf{s}$, the momentum density, and hence the velocity, $\mathbf{V}(\mathbf{r}, t) \equiv \frac{1}{n(\mathbf{r},t)} \int \mathbf{v}f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t)d\mathbf{v}d\mathbf{s}$, are hydrodynamic fields. Let $\mathbf{u} \equiv \mathbf{v} - \mathbf{V}(\mathbf{r}, t)$ denote the peculiar velocity. The (translational) granular temperature, $\mathbf{T}(\mathbf{r}, t) \equiv \frac{1}{n(\mathbf{r},t)} \int u^2 f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t)d\mathbf{v}d\mathbf{s}$ is nearly conserved in the near-elastic, nearly smooth case, and is therefore a hydrodynamic field. In the smooth *limit* the spin is decoupled from the translational degrees of freedom; hence the number density corresponding to each value of the spin, $n(\mathbf{s}, \mathbf{r}, t) \equiv \int f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t)d\mathbf{v}$, is conserved as well, as is the (more convenient to use) relative spin distribution field, $\rho(\mathbf{s}, \mathbf{r}, t) \equiv \frac{n(\mathbf{s},\mathbf{r},t)}{n(\mathbf{r},t)}$. Hence the infinite set of relative spin dependent number densities, $\rho(\mathbf{s}, \mathbf{r}, t)$, qualify as hydrodynamic fields. In other words, the system is considered to be a mixture, whose "components" are indexed by their respective values of the spin. A field of interest is the velocity field corresponding to particles of spin $\mathbf{s}$: $\mathbf{U}(\mathbf{s}, \mathbf{r}, t) = \frac{1}{n(\mathbf{s},\mathbf{r},t)} \int \mathbf{v}f(\mathbf{v}, \mathbf{s}, \mathbf{r}, t)d\mathbf{v}$. The latter field is not hydrodynamic, as it does not correspond to a conserved entity, and, indeed, it is enslaved to the slow fields, i.e., expressible in terms of them (see below). A standard procedure now produces the continuum equations of motion from the Boltzmann equation:

$$n\frac{DV_\alpha}{Dt} + \frac{\partial}{\partial r_\beta}P_{\alpha\beta} = 0,$$

$$n\frac{DT}{Dt} + 2\frac{\partial V_\alpha}{\partial r_\beta}P_{\alpha\beta} + 2\frac{\partial Q_\alpha}{\partial r_\alpha} = -2\Gamma,$$

$$n\frac{D\rho(\mathbf{s})}{Dt} = \int B(f, f, \mathbf{v}, \mathbf{s})d\mathbf{v}$$
$$\qquad\qquad - \nabla \cdot (n\rho(\mathbf{s})\delta\mathbf{V}(\mathbf{s})), \qquad (5)$$

where $\frac{D}{Dt}$ is the material derivative, $\delta\mathbf{V}(\mathbf{s}, \mathbf{r}, t) \equiv \mathbf{U}(\mathbf{s}, \mathbf{r}, t) - \mathbf{V}(\mathbf{r}, t)$ denotes the relative velocity between

particles of spin $\mathbf{s}$ and the hydrodynamic velocity, and the summation convention is assumed. The stress tensor is given by $P_{\alpha\beta} = \int u_\alpha u_\beta f d\mathbf{u}d\mathbf{s}$, the heat flux vector is given by $Q_\alpha = \frac{1}{2} \int u_\alpha u^2 f d\mathbf{u}d\mathbf{s}$, and the energy sink term is $\Gamma = - \int d\mathbf{v}_1 d\mathbf{s}_1 \frac{v_1^2}{2} B(f, f, \mathbf{v}_1, \mathbf{s}_1)$.

In this Letter we consider the (nearly smooth) case $\beta_0 \approx -1$ and small friction μ_f. We further specialize, for convenience, to the case $\gamma_0 \approx \frac{\pi}{2}$, which corresponds to $\frac{1+\beta_0}{\mu_f} < 1$. In physical terms, this means that β is taken to be nearly constant (as in some models [4,10]), except at near grazing collisions. In this case, μ_f affects the results only to third order in perturbation theory (not presented here). The small parameters employed in the perturbative solution of the Boltzmann equation are therefore $\epsilon \equiv 1 - e^2$ (the degree of inelasticity), $\epsilon_3 \equiv 1 - \beta_0^2$ (the smoothness parameter), and the Knudsen number, $K \equiv \frac{\ell}{L}$ ($\ell = \frac{1}{\pi n d^2}$ is the mean free path, n the number density, and L a macroscopic scale), which is a measure of the gradients.

The zeroth order solution ($\epsilon = \epsilon_3 = K = 0$) of the Boltzmann equation reads $f_0(\mathbf{u}, \mathbf{s}) = f_M(\mathbf{u})\rho(\mathbf{s})$, where $f_M(\mathbf{u}) = n(\frac{3}{2\pi T})^{3/2}e^{-3u^2/2T}$, and $\rho(\mathbf{s})$ is practically any normalizable distribution, since at this order the spins are decoupled from translation. Next, let $f(\mathbf{v}, \mathbf{s}) = f_0(\mathbf{u}, \mathbf{s}) \times [1 + \phi(\mathbf{u}, \mathbf{s})]$, and expand ϕ as follows: $\phi = K\phi_K + \epsilon\phi_\epsilon + \epsilon_3\phi_3 + K\epsilon\phi_{K\epsilon} + \cdots$, where, e.g., ϕ_ϵ is the $\mathcal{O}(\epsilon)$ term in the expansion.

The perturbative solution of the Boltzmann equation, in powers of K, ϵ, and ϵ_3, involves the repeated solution of $\tilde{L}\phi_{\text{given order}} = R_{\text{same order}}$, where $R_{\text{same order}}$ depends on previous orders, and the linearized Boltzmann operator, $\tilde{L}$, is given by $\tilde{L}\phi \equiv d^2 \int_{\mathbf{k}\cdot\mathbf{v}_{12}>0} d\mathbf{v}_2 d\mathbf{s}_2 d\mathbf{k}(\mathbf{k} \cdot \mathbf{v}_{12})f_0(\phi_1' + \phi_2' - \phi_1 - \phi_2)$. The heavy algebra involved in these calculations has been performed using a symbolic processor. The resulting constitutive relations, up to $\mathcal{O}(\epsilon^2, \epsilon_3^2, \epsilon\epsilon_3, K\epsilon, K\epsilon_3)$, are presented next [the $O(K^2)$ terms are the same as in the smooth case [17]]. The stress tensor is given by

$$P_{\alpha\beta} = \frac{1}{3}nT\delta_{\alpha\beta} - 2n\ell\mu\sqrt{T}\overline{\frac{\partial V_\alpha}{\partial r_\beta}} - \frac{n\epsilon_3^2}{(1+\tilde{I})^2}\left(0.0129\tilde{I}^2 \int \overline{\mathbf{s}_{12_\alpha}\mathbf{s}_{12_\beta}}\rho(\mathbf{s}_1)\rho(\mathbf{s}_2)d\mathbf{s}_1 d\mathbf{s}_2 - 0.000599T\overline{B}_{\alpha\beta}\right) \qquad (6)$$

where $\delta_{\alpha\beta}$ is Kronecker's delta, $\mu = 0.325 + 0.0577\epsilon^*$, with $\epsilon^* = \epsilon + \epsilon_3\frac{\tilde{I}}{1+\tilde{I}}$, and $B_{\alpha\beta} = \int \rho(\mathbf{s})\Sigma_{\alpha\beta}\Sigma_{\gamma\gamma}d\mathbf{s}$, where $\Sigma_{\alpha\beta} = \frac{1}{\rho(\mathbf{s})}\frac{\partial}{\partial s_\alpha}(\rho(\mathbf{s})(\mathbf{S} + \mathbf{s})_\beta)$. The spin density is $\mathbf{S}(\mathbf{r}, t) \equiv \int \mathbf{s}\rho(\mathbf{s}, \mathbf{r}, t)d\mathbf{s}$, and $\overline{\mathbf{A}}$ denotes the traceless symmetric part of the tensor $\mathbf{A}$. The heat flux is given by $Q_\alpha = -2\kappa n\ell\sqrt{T}\frac{\partial T}{\partial r_\alpha} - 2\lambda\ell T^{3/2}\frac{\partial n}{\partial r_\alpha}$, where $\kappa = 0.205 + 0.0536\epsilon^*$, and $\lambda = 0.1054\epsilon^*$. The energy sink term is given by $\Gamma = \frac{n}{\ell}\sqrt{T}(\alpha T - \beta T_{\text{rot}} - 2\beta S^2)$, where $\alpha = \sqrt{\frac{4}{27\pi}}[\epsilon^* - 0.02526\epsilon^{*2}] + \frac{\beta}{\tilde{I}}$, $\beta = \sqrt{\frac{1}{108\pi}}(\frac{\tilde{I}}{1+\tilde{I}})^2\epsilon_3^2$, and $T_{\text{rot}}(\mathbf{r}, t) \equiv \int [\mathbf{s} - \mathbf{S}(\mathbf{r}, t)]^2\rho(\mathbf{s}, \mathbf{r}, t)d\mathbf{s}$ is the rotational temperature. Note that $\mathbf{S}$ and T_{rot} are not independent hydrodynamic fields, since they are moments of the hydrodynamic fields, $\rho(\mathbf{s}, \mathbf{r}, t)$. Next, $\delta\mathbf{V}$, which appears in the equation of motion for $\rho(\mathbf{s})$, is given by

$$\delta V_\alpha(\mathbf{s}) = -\frac{\ell\sqrt{T}}{10}\left(4.79 + 2.06\epsilon + \frac{0.248\tilde{I}}{1+\tilde{I}}\epsilon_3\right)\frac{\partial \log\rho(\mathbf{s})}{\partial r_\alpha} + \frac{\epsilon_3\ell\sqrt{T}}{100(1+\tilde{I})}\left\{(5.06\Sigma_{\gamma\gamma}\delta_{\alpha\beta} + 0.43\overline{\Sigma}_{\alpha\beta})\frac{\partial \log T}{\partial r_\beta}\right.$$

$$\left. + (8.30\Sigma_{\gamma\gamma}\delta_{\alpha\beta} + 0.606\overline{\Sigma}_{\alpha\beta})\frac{\partial \log n}{\partial r_\beta} + \frac{1}{\rho(\mathbf{s})}\frac{\partial[\rho(\mathbf{s})(3.93\Sigma_{\gamma\gamma}\delta_{\alpha\beta} + 0.738\overline{\Sigma}_{\alpha\beta})]}{\partial r_\beta}\right\}. \qquad (7)$$

Notice that $\delta \mathbf{V}(\mathbf{s}) \neq 0$ due to spatial gradients. The term $\int B d\mathbf{v}$ in Eq. (5) is not presented here in full for sake of brevity. It can be expressed as follows:

$$\int B(f, f, \mathbf{v}, \mathbf{s}) d\mathbf{v} \equiv \frac{\epsilon_3}{(1 + \tilde{I})^2} \frac{n}{\ell} \sqrt{T} \nabla_{\mathbf{s}} \cdot \mathbf{G}(\mathbf{s}), \quad (8)$$

where $\mathbf{G}$ is specified below for a special case. It follows from Eqs. (5), (7), and (8) that the spin number densities satisfy a (generalized) diffusion equation, as expected for a "mixture." Note that to linear order in ϵ_3 the constitutive relations correspond to smooth grains with an effective coefficient of normal restitution, as in [11]. This property does not hold at higher orders in ϵ_3 [which are often consequential; see, e.g., Eq. (9)].

Consider first the homogeneous cooling state. Here all spatial derivatives vanish, as does the velocity, so that the second and third of Eqs. (5) read, respectively, $n\frac{dT}{d\tau} = -2\frac{\ell}{\sqrt{T}}\Gamma$ and $\frac{d\rho(\mathbf{s})}{d\tau} = \frac{\epsilon_3}{(1+\tilde{I})^2}\nabla_{\mathbf{s}} \cdot \mathbf{G}(\mathbf{s})$, where $\tau \equiv \int_0^t \frac{T^{1/2}}{\ell} dt'$ represents the number of accumulated collisions per particle; cf., e.g., [4]. It now follows that

$$\frac{dT}{d\tau} = -2\alpha T + 2\beta T_{\mathrm{rot}} + 4\beta S^2,$$

$$\frac{dT_{\mathrm{rot}}}{d\tau} = \nu T - 2\eta T_{\mathrm{rot}} + \chi S^2, \qquad \frac{dS^2}{d\tau} = -\delta S^2, \quad (9)$$

where $\eta = \frac{\epsilon_3}{(1+\tilde{I})^2}(d^{(2)} - 0.207\epsilon_3)$, $\nu = 0.109\frac{\epsilon_3^2}{(1+\tilde{I})^2}$, $\chi = 0.369\frac{\epsilon_3^2}{(1+\tilde{I})^2}$, $\delta = \frac{\epsilon_3}{(1+\tilde{I})^2}(4d^{(2)} - 0.467\epsilon_3)$, and $d^{(2)} = \sqrt{\frac{4}{3\pi}}(1+\tilde{I})(\frac{1}{3} + 0.00333\epsilon + \frac{0.619\tilde{I}+2.55}{10(\tilde{I}+1)}\epsilon_3)$. Equation (9) agrees with the mean field result of [6]. The three eigenvalues of Eq. (9) are negative. The long-time decay rate of T and T_{rot} follows Haff's law. Also, $\lim_{t\to\infty}\frac{S^2}{T} = 0$ and $\lim_{t\to\infty}\frac{T_{\mathrm{rot}}}{T} = r$, where $r = \frac{1}{2\beta}[\alpha - \eta + \sqrt{(\alpha-\eta)^2 + 2\nu\beta}]$. Since S^2 decays to zero faster than T and T_{rot}, it is justified to consider the asymptotic time dynamics for the case $\mathbf{S} = 0$. Next, multiplying the equation satisfied by ρ and by $s_i s_j$ and integrating over $\mathbf{s}$ yields $\lim_{t\to\infty}\frac{1}{T}\int s_i s_j \rho(\mathbf{s})d\mathbf{s} = r\delta_{ij}$, indicating that $\rho(\mathbf{s})$ is isotropic. When $\rho(\mathbf{s})$ depends on s alone, the function $\mathbf{G}$ (whose dependence on the fields is not presented above) is given by $G_i = \mathcal{G}(s)s_i$, where

$$\mathcal{G}(s) = \left(A\rho + \frac{\epsilon_3}{s}(BT + Cs^2)\frac{d\rho}{ds}\right), \quad (10)$$

where $A = d^{(2)} + 0.00738\epsilon_3$, $B = (1.81 + 1.45\frac{T_{\mathrm{rot}}}{T}) \times 10^{-2}$, and $C = 3.41 \times 10^{-2}$. The equation satisfied by $\rho(\mathbf{s})$ possesses a scaling solution of the form $4\pi s^2 \rho(s, \tau) = \frac{1}{\sqrt{T}}F(\zeta)$ where $\zeta = \frac{s}{\sqrt{\epsilon_3 T}}$, and $F(\zeta)$ satisfies

$$F_{\zeta\zeta} + \frac{F_\zeta}{\zeta}\frac{\epsilon_3 DC\zeta^2 - 2B}{\epsilon_3 C\zeta^2 + B} + \frac{F}{\zeta^2}\frac{\epsilon_3(D-2)C\zeta^2 + 2B}{\epsilon_3 C\zeta^2 + B} = 0, \quad (11)$$

where $D \equiv \frac{\epsilon_3 A - (1+\tilde{I})^2(\alpha-\beta r)}{\epsilon_3^2 C}$. The normalizable solution of Eq. (11) satisfies (at long times) $F(\zeta) \sim \zeta^{2-D}$ for large ζ;
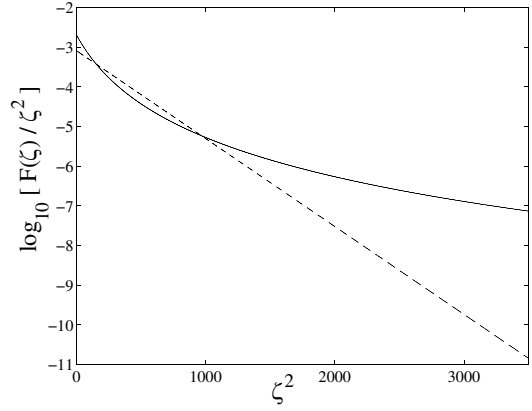

FIG. 1. The rescaled spin distribution, $F(\zeta)$, for the homogeneous cooling state (full line) and a Gaussian distribution (dashed line) with the same value of T_{rot}/T, for $\epsilon = \epsilon_3 = 0.1$. Recall that the rescaled spin ζ equals $s/\sqrt{\epsilon_3 T}$.

i.e., it decays algebraically and, hence, $\rho(s) \sim \frac{1}{T^{3/2}}(\frac{s}{\sqrt{T}})^{-D}$. Figure 1 compares this solution with a Gaussian distribution for $\epsilon = \epsilon_3 = 0.1$.

Next consider the general case of a homogeneous steady state. In this case [see Eq. (7)] $\delta\mathbf{V}(\mathbf{s}) = \mathbf{0}$, and Eq. (5) implies $\nabla_{\mathbf{s}} \cdot \mathbf{G} = \mathbf{0}$. Multiplying $\nabla_{\mathbf{s}} \cdot \mathbf{G}$ by s_i and $s_i s_j$, and integrating over the spin, yields $\mathbf{S} = \mathbf{0}$ and $\int s_i s_j \rho(\mathbf{s})d\mathbf{s} \propto \delta_{ij}$, respectively, indicating, again, that $\rho(\mathbf{s})$ is isotropic. Thus $G_i = \mathcal{G}(s)s_i$, as given by Eq. (10). Using again the equation $\nabla_{\mathbf{s}} \cdot \mathbf{G} = \mathbf{0}$, with $G_i = \mathcal{G}(s)s_i$, we obtain $\frac{\partial}{\partial s} \times (s^3\mathcal{G}) = 0$. The only normalizable solution of this equation is $\mathcal{G} = 0$. It now follows from Eq. (10) that $\rho(\zeta) = A_0(B + \epsilon_3 C\zeta^2)^{-A/2\epsilon_3 C}$, where A_0 is a normalization constant; hence the spin distribution decays algebraically for any homogeneous steady state. Also, in this state $\frac{T_{\mathrm{rot}}}{T} = \frac{0.0543\epsilon_3}{A - 0.214\epsilon_3}$.
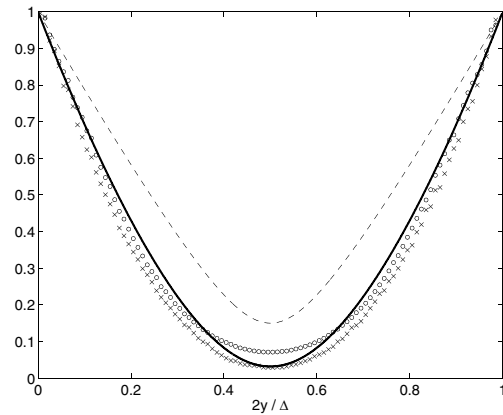

FIG. 2. $\frac{T}{T_B}$ (dashed line) and $\frac{S_z}{S_z(B)}$ (full line) in a wall-bounded shear flow, with $\epsilon = \epsilon_3 = 0.1$, $\frac{S_z(B)}{\nu_0}\frac{d}{\Delta} = 0.2$, $\phi = 0.15$, and $\Phi = 0.01$, and results of MD simulations (crosses for spin, circles for temperature) of 131 072 spheres with the same parameters (and rough walls). The MD temperature profile is slightly below the theoretical prediction (and exhibits a "plug"—see [1] for the possible mechanism); this seems to be a finite density effect.
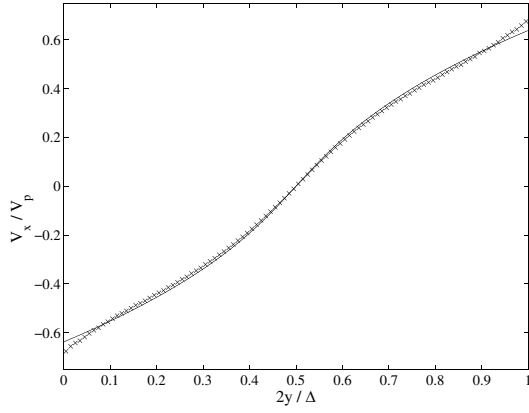
FIG. 3.   Theoretical profile (full line) and MD results (crosses) for $\frac{V_x}{V_p}$ for the same flow as in Fig. 2.

Consider next a shear flow confined between two parallel plates located at $y = \pm\frac{\Delta}{2}$ moving in the x direction with velocities $\pm V_p$, respectively. Let the hydrodynamic fields depend on y alone and $\mathbf{V} = V_x(y)\hat{\mathbf{x}}$. The following boundary conditions are employed [18–20]: $P_{xy(B)} = -\frac{\sqrt{3}\pi}{6}\phi n_B\sqrt{T_B}v_s$, $Q_{y(B)} = \mp\frac{\sqrt{3}\pi}{6}(\phi v_s^2 - \frac{3}{2}\Phi T_B)n_B\sqrt{T_B}$, and $S_{z(B)} = -\frac{d}{2}\frac{dV_x}{dy}$ where $v_s \equiv V_p - V_x(\frac{\Delta}{2})$ is the slip velocity, ϕ is the wall roughness coefficient, Φ is the rate of energy loss at the boundaries, and the subscript B denotes values at the boundaries. The average number density is taken to be $\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} n(y)dy = n_0$ (i.e., the mean volume fraction is $\nu_0 = \frac{\pi}{6}n_0 d^3$). It is convenient to define a new spatial variable: $\xi \equiv \frac{2\sqrt{T_B}}{\Delta}\int_0^y \frac{dy'}{\sqrt{T(y')}}$. To first order in ϵ_3 the first and second of Eqs. (5) are spin independent. Their solution is $p_0 = \frac{1}{3}nT$, $V_x = C_1\sqrt{T_B}\xi$, and $T = B_2 T_B \xi^2 + T_0$, where p_0, C_1, B_2, and T_0 are constants fixed by the boundary conditions. Multiplying the third of Eqs. (5) by s_z, and integrating over the spin, yields

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{S_z}{\sqrt{T}}\right) - \epsilon_3 C_a\left(1 - \frac{T_0}{T}\right)\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\frac{S_z}{\sqrt{T}}\right)$$

$$+ \epsilon_3\left[\frac{T_B}{T}\left(\frac{n_0\Delta}{n\ell}\right)^2\left(\frac{p_0}{T_B n_0}\right)^2 C_b\right.$$

$$\left. + C_a\left(1 - \frac{T_0}{T}\right)\left(1 - 2\frac{T_0}{T}\right)\frac{1}{\xi^2}\right]\left(\frac{S_z}{\sqrt{T}}\right) = 0, \quad (12)$$

where $C_a = 1.274/C_t$, $C_b = 45(1 + \tilde{I})\delta/(\epsilon_3 C_t)$ with $C_t = (4.79 + 2.06\epsilon)(1 + \tilde{I}) + (0.7358 + 0.248\tilde{I})\epsilon_3$. The theoretical results (in terms of the physical coordinate, y) are presented in Figs. 2 and 3 alongside a comparison with molecular dynamics (MD) simulations. An analysis of Eq. (12) reveals that the spin produced at the boundaries

decays as a function of the distance from the boundary, the typical penetration depth being of the order of $\frac{\ell}{\sqrt{\epsilon_3}}$. Also notice (Fig. 3) the shear boundary layer, which was also seen in other simulations [18].

Finally, as only the dilute case is considered here, the stress tensor (being the average of a symmetric entity) is symmetric; i.e., one does not obtain a micropolar theory [21].

*Electronic address: isaac@eng.tau.ac.il
†Electronic address: henri@eng.tau.ac.il
‡Electronic address: fnbarlev@post.tau.ac.il

[1] I. Goldhirsch, Annu. Rev. Fluid Mech. **35**, 267 (2003).
[2] S. J. Moon, J. B. Swift, and H. L. Swinney, Phys. Rev. E **69**, 031301 (2004).
[3] S. Nasuno, A. Kudrolli, and J. P. Gollub, Phys. Rev. Lett. **79**, 949 (1997).
[4] S. Luding, M. Huthmann, S. McNamara, and A. Zippelius, Phys. Rev. E **58**, 3416 (1998).
[5] M. Huthmann and A. Zippelius, Phys. Rev. E **56**, R6275 (1997).
[6] O. Herbst, M. Huthmann, and A. Zippelius, Granular Matter **2**, 211 (2000).
[7] S. Luding, Phys. Rev. E **52**, 4442 (1995).
[8] S. McNamara and S. Luding, Phys. Rev. E **58**, 2247 (1998).
[9] J. T. Jenkins and M. W. Richman, Phys. Fluids **28**, 3485 (1985).
[10] C. K. K. Lun, J. Fluid Mech. **233**, 539 (1991).
[11] J. T. Jenkins and C. Zhang, Phys. Fluids **14**, 1228 (2002).
[12] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, London, 1970).
[13] J. S. Dahler and M. Theodosopulu, Adv. Chem. Phys. **31**, 155 (1975).
[14] I. G. Shukhman, Astron. Zh. **61**, 985 (1984); S. Araki, Icarus **76**, 182 (1988).
[15] N. Maw, J. R. Barber, and J. N. Fawcett, Trans. ASME J. Lubr. Technol. **103**, 74 (1981).
[16] H. Kuninaka and H. Hayakawa, Phys. Rev. Lett. **93**, 154301 (2004).
[17] N. Sela and I. Goldhirsch, J. Fluid Mech. **361**, 41 (1998).
[18] P. R. Nott *et al.*, J. Fluid Mech. **397**, 203 (1999).
[19] N. Mitarai and H. Nakanishi, J. Fluid Mech. **507**, 309 (2004).
[20] J. T. Jenkins, Trans. ASME J. Appl. Mech. **59**, 120 (1992).
[21] N. Mitarai, H. Hayakawa, and H. Nakanishi, Phys. Rev. Lett. **88**, 174301 (2002).