Nonlinear Saturation of Tearing Mode Islands

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New, rigorous results for the tearing island saturation problem are presented. These results are valid for the realistic case where the magnetic island structure is nonsymmetric about the reconnection surface and the electron temperature, on which the electrical resistivity depends, is evolved self-consistently with the island growth.

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Magnetic reconnection is a ubiquitous phenomenon in magnetically confined plasmas, both in space and in laboratory experiments. This process causes a topological transition of the magnetic field configuration, accompanied by a transformation of magnetic energy into plasma kinetic energy and heat in a relatively short time. In magnetic fusion experiments, reconnection often occurs spontaneously and this classical instability is well known as tearing modes [1]. In tokamak devices, these modes are often responsible for degraded plasma confinement and are seen as a potential threat for the successful operation of burning plasma experiments.

There is a large body of literature on the theory of resistive tearing modes (see, e.g., Ref. [2]). Linear stability investigations were carried out initially in slab geometry [1], and later extended to cylindrical [3] and to full toroidal geometry [4]. Analytic investigations of the nonlinear phase of tearing modes have been fewer. Rutherford [5] developed a theory of the nonlinear evolution and showed that the exponential growth of linear theory is replaced by a growth phase in which the width, W, of the reconnected island is governed by the simple equation $dW/d\tau_\eta = 1.22\Delta'$, where $W = 4[\psi_0/L^2\psi''_{\mathrm{eq}*}(0)]^{1/2}$ is the dimensionless island width, $\psi_{\mathrm{eq}*}(0)$ is the equilibrium helical magnetic flux function at the reconnecting surface, ψ_0 is the value of the perturbed helical magnetic flux at the island X-point, L is a convenient normalization scale, $\tau_\eta = \eta t/L^2$ is time on the resistive diffusion scale, $\Delta' = \lim_{\epsilon\to 0}(L/\psi_0)(d\psi_{\mathrm{out}}/dx)|_{-\epsilon}^{+\epsilon}$ is the linear stability index [1], and ψ_{out} is the outer perturbed flux.

The first analytic study of the nonlinear saturation of tearing modes in the low-β (= kinetic pressure/magnetic pressure) approximation was presented by White *et al.* [6]. In this pioneering work, a quasilinear treatment was adopted, such that only the dominant Fourier component of the linear mode structure was assumed to play a role in the nonlinear saturation process. An *ansatz* for the nonlinear island structure was assumed. As a consequence, the analytic results obtained in [6] are approximate and indeed Biskamp noted a disagreement with numerical results [2]. In Ref. [7], Thyagaraja established a systematic expansion procedure, assuming $\ln(1/W) \gg 1$, so that terms of order $O(W)$ were neglected relative to terms $O(W\ln(1/W))$. A rigorous result was also obtained in Refs. [8,9] for slab geometry models where the equilibrium current density, J_{eq}, is symmetric about the reconnecting surface. Further results relevant to the present discussion were obtained in [10,11]. But approximations for the island structure were still adopted in these papers, and as a consequence only part of the answer was obtained [see the discussion below Eq. (8)]. In addition, all the foregoing work dealt with situations where the electrical resistivity was assumed to be unaffected by the island formation.

In this Letter, we are interested in the realistic case where the equilibrium current density is not symmetric about the reconnecting surface, and the dependence of the resistivity profile on the plasma temperature, T, is accounted for. For simplicity, we adopt the classical (Spitzer) expression for resistivity in a fully ionized plasma, $\eta \propto T^{-3/2}$ and neglect complications such as the dependence of η on the effective charge in an impure tokamak plasma. In a practical tokamak equilibrium configuration, $\eta_{\mathrm{eq}}J_{\mathrm{eq}}$ is balanced by a constant toroidal electric field, E. Therefore, since J_{eq} has a nonconstant radial profile, so does η. As the saturated magnetic island exceeds a critical width, $W_c \sim (\kappa_\perp/\kappa_\parallel)^{1/4}$, determined by the ratio of perpendicular over parallel thermal conductivity [12], the temperature profile must relax; in particular, it should become nearly constant within the island region on account of the very large thermal conductivity along the magnetic field lines. Therefore, as already pointed out by Rutherford [5], the treatment of the island saturation problem requires an equation for the plasma temperature evolution.

Our mathematical discussion is organized as follows. We provide a somewhat detailed derivation of the island saturation problem in slab geometry. The case of small magnetic islands, $W \ll W_c$, is treated first. In this case, the equilibrium resistivity profile is not modified by the change in magnetic topology and we obtain an island saturation equation, which extends and corrects the results reviewed above. Then, we consider the case $W \gg W_c$. Finally, results in the cylindrical case are summarized (our procedure applies to modes with poloidal mode number $m > 1$).

In slab geometry, we assume a magnetic field configuration of the type $\mathbf{B} = B_0\mathbf{e_z} + \nabla\Psi \times \mathbf{e_z}$, where $B_0 =$ const and $\Psi(x, \xi) = \psi_{eq}(x) + \psi(x, \xi)$, with $\xi = k_y y + k_z z$. In order to mimic toroidal geometry, y and z are assumed to be periodic coordinates, so that the wave vector components k_y and k_z acquire discrete values. It is convenient to write $\mathbf{B} = \mathbf{B_C} + \mathbf{B}_*$, where $\mathbf{k} \cdot \mathbf{B_C} = 0$ everywhere and $\mathbf{B}_* = \nabla\Psi_* \times \mathbf{e_z}$ is the sheared magnetic field, which is the part of $\mathbf{B}$ involved in the reconnection process; $\Psi_* = \Psi - k_z B_0 x/k_y$ has the meaning of *helical* flux. Magnetic flux surfaces correspond to $\Psi_* =$ const. In slab geometry, the helical current density, $J_* = -\nabla_\perp^2 \Psi_*$, coincides with the current density along the z direction, J, where the operator $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$. The resonant reconnecting surface is located at $x = 0$, where $d\psi_{*\mathrm{eq}}/dx$ vanishes. The problem is assumed to be two-dimensional, with the coordinate orthogonal to x and ξ as the ignorable coordinate. The nonlinear problem involves a single helicity, k_z/k_y; however, several harmonics in ξ are required in order to account for the nonsymmetric structure of the magnetic island. The plasma flow is $\mathbf{v} = v_z\mathbf{e_z} + \mathbf{e_z} \times \nabla\phi$, with ϕ the stream function. Following a standard procedure [5], the resistive MHD model can be reduced to two equations for Ψ_* and ϕ. The first equation is the plasma vorticity equation:

$$\partial_t U + [\phi, U] = [J, \Psi_*], \tag{1}$$

where $U = \nabla_\perp^2 \phi$ is the vorticity component along the z direction, $[A, B] = \mathbf{e_z} \cdot \nabla A \times \nabla B$, and a constant density has been assumed. The second equation is the z component of the resistive Ohm law:

$$\partial_t \Psi_* + [\phi, \Psi_*] = E - \eta(T)J. \tag{2}$$

For the small island case, $W \ll W_c$, the temperature profile is not modified by the presence of the island, so we can take $T = T_{\mathrm{eq}} \propto J_{\mathrm{eq}}^{2/3}$ and $\eta(T) = \eta_{\mathrm{eq}}(x)$ in Eq. (2). At saturation, the ∂_t terms vanish. Inspection of Eq. (2) reveals that ϕ is $O(\eta)$ as compared to Ψ_*, hence the inertial term $[\phi, U]$ can be neglected in Eq. (1), which then yields $J = J(\Psi_*)$ to lowest order in η. The term $[\phi, \Psi_*]$ in Eq. (2) is annihilated by flux surface averaging, yielding

$$E = \langle \eta(T_{\mathrm{eq}})J \rangle, \tag{3}$$

where $\langle f \rangle = (\int |\frac{\partial\Omega}{\partial X}|^{-1} f d\xi)/(\int |\frac{\partial\Omega}{\partial X}|^{-1} d\xi)$ denotes the flux surface average of f. These equations govern the small island calculation [5,7,12], with $\Omega = [\Psi_*(X, \xi) - \psi_{\mathrm{eq}}(0)]/\psi_0$ a conveniently normalized flux variable and $X = x/W$.

Following Ref. [7], we employ an expansion of $J(\Omega)$ in powers of W, $J(\Omega) = J_0(\Omega) + WJ_1(\Omega) + W^2 J_2(\Omega) + O(W^3)$. A similar expansion is assumed for $\Omega(X, \xi)$. Using the equilibrium relation $\eta_{\mathrm{eq}} = E/J_{\mathrm{eq}}$, we find $J(\Omega) = \langle J_{\mathrm{eq}}^{-1} \rangle^{-1}$. Expanding $J_{\mathrm{eq}}(x) = J_{\mathrm{eq}0}$, we obtain up to terms $O(W^2)$:

$$\frac{J(\Omega)}{J_{\mathrm{eq}0}} = 1 + Wa\langle X \rangle + W^2\left(\left(\frac{b}{2} - a^2\right)\langle X^2 \rangle + a^2 \langle X \rangle^2\right). \tag{4}$$

The difficulty in the nonlinear problem lies in the fact that the shapes of the modified magnetic surfaces, which are needed in order to evaluate the flux surface averages, are not known initially. However, Thyagaraja [7] showed that the solution for $J(\Omega)$ to any given order in the W expansion, only requires knowledge of the flux surface shape, i.e., of Ω, to one order less in the expansion. Thus, starting from the lowest order, symmetric, expression [5,12] for $\Omega_0 = 8X^2 - \cos\xi$, one finds, $J_1 = J_0 a\langle X \rangle$, which vanishes within the island ($-1 < \Omega_0 < 1$) and is finite outside the separatrix. And, crucially, this expression for J_1 only requires the flux surface average $\langle X \rangle_0$ calculated using the symmetric island structure defined by $\Omega_0(X, \xi)$. The corrections arising from a more accurate specification of $\Omega = \Omega_0 + W\Omega_1$ are $O(W)$ smaller and are recovered in next order of the Thyagaraja expansion. A similar procedure has been followed recently by Escande *et al.* [13].

The Ampere equation then takes the form

$$\frac{\partial^2 \Omega_1}{\partial X^2} = \begin{cases} \frac{a\sigma\pi}{4K(m)\sqrt{m}}, & \Omega_0 > 1 \\ 0, & |\Omega_0| < 1 \end{cases} \tag{5}$$

where $\sigma = X/|X|$, K is the elliptic integral of the first kind and $m = 2/(\Omega_0 + 1)$. Equation (5) is used to determine Ω_1. Then, $J_2(\Omega)$ is calculated, requiring the evaluation of $\langle X \rangle$ to $O(W)$, taking account of the asymmetric flux surfaces defined by $\Omega = \Omega_0 + W\Omega_1$. Integrating up Eq. (5) in the three regions (outside the separatrix, $X < 0$ and $X > 0$, and within the island), introduces six arbitrary functions, $C_j(\xi)$. Continuity of $\partial_X \Omega$ and Ω at the separatrix reduces these to two, which are then determined by matching to the linear eigenmode in the outer region. Surprisingly, this leaves a part of Ω_1 (*inner*) which does not match to either the equilibrium flux in the outer region or the linear perturbed flux. The unmatched part of Ω_1 is of odd parity in σ and is constant ($\sim 0.37\sigma$) in the $X \to \infty$ limit. When expressed in global outer variables it is of magnitude $O(W^3)$, relative to the equilibrium. It therefore appears that reconnection in the inner region is responsible for driving a nonlinear modification of the outer equilibrium. But since this is $O(W^3)$, it will have a negligible effect on the stability index Δ' and hence on mode saturation.

The final equation, relating Δ' to the island width W, is obtained by evaluating the jump in the logarithmic derivative, $(\partial_x \Omega)/\Omega$, across the "inner region" where the magnetic topology has changed. Thus, asymptotic matching of the quantities M_{in} and M_{out},

$$M_{\mathrm{in}} = \frac{-16}{WJ_{0*}\pi} \int_{-X}^{X} dX' \int_{-\pi}^{\pi} d\xi \left(J + \frac{1}{16}\frac{\partial^2 \Omega}{\partial \xi^2}\right)\cos(\xi), \tag{6}$$

$$M_{\rm out} = \frac{L}{\Omega_{\rm out}(0)}\left(\frac{\partial\Omega_{\rm out}}{\partial x}\bigg|_{+WX} - \frac{\partial\Omega_{\rm out}}{\partial x}\bigg|_{-WX}\right), \qquad (7)$$

yields the island saturation equation

$$\Delta' = 0.41W\{a^2[\ln(1/W) + 4.85] - aA/2 - b\}, \qquad (8)$$

where $a = LJ'_{\rm eq}/J_{\rm eq}$ and $b = L^2 J''_{\rm eq}/J_{\rm eq}$, both evaluated at the reconnecting surface, while A is a measure of the asymmetry of the global (linear) eigenmode [14]. Note that the combination $\ln(1/W) - 0.5A/a$ does not depend on the choice of the normalization length, L, and therefore it is scale invariant. Equation (8) determines the amplitude of the saturated island in the $W \ll W_c$ limit. This equation correctly reduces to the symmetric limit [8,9] where $a = 0$.

As compared with previous results, we note that: White *et al.* [6] had both the $\ln(W)$ and the A terms, although multiplied by different numerical coefficients; Thyagaraja [7] obtained the correct form of the $\ln(W)$ term, but neglected the other terms (as a consequence the result in [7] is not scale invariant and therefore physically questionable); Zakharov *et al.* [10] had only the A term, in addition to a finite β effect, which we have not taken into account in our work; Pletzer and Perkins [11] had both the $\ln(W)$ and the A terms with the same numerical factors as in (8), but missed the other terms.

The large island calculation, $W \gg W_c$, follows similar lines to that of the small island, but since $J(\Omega) = J_0[T(\Omega)/T_0]^{3/2}$, the electron temperature $T(\Omega)$ must be obtained by integrating up the constraint equation [5,12]

$$\frac{dT}{d\Omega}\int \kappa_\perp(x)|\nabla\Omega| d\xi = W\kappa_{\perp 0}T'_0/16, \qquad (9)$$

obtained by expansion of $\nabla\cdot\mathbf{q}_e = 0$ in the $\kappa_\perp/\kappa_\parallel \ll 1$ limit, where $\mathbf{q}_e$ is the electron heat flux. For the sake of simplicity, it is assumed that heat sources and sinks can be neglected locally, while their presence far away from the island is taken into account by appropriate boundary conditions. The right hand side of Eq. (9) has been chosen to match the heat flux to its outer region value. With a heat source for $X \to -\infty$ and a constant temperature boundary condition at $X \to +\infty$, two cases are now considered, corresponding to different physical conditions. In the first case, which we report here, the global thermal equilibrium is unaltered, and steep temperature gradients appear in boundary layers at the separatrix of the island. Thus, the inner temperature is matched to the original equilibrium $T_{\rm eq}(x)$ on both sides of the island. This corresponds to calculating at an early time on the thermal transport time scale. In the second case (to be described in a future paper), the steep electron temperature gradients have relaxed, T is continuous across the separatrix and the core temperature has dropped, i.e., degradation of core confinement has taken effect [15]. In this case the current density in the core ($X < 0$) is reduced, the inductive electric field will have increased to maintain constant plasma current I, thus increasing current density for $X > 0$. These perturbations of the global equilibrium modify the original drive, Δ', so that a more detailed knowledge of the equilibrium is required to make predictions in this case.

For the large island calculation, the temperature is expanded in the form, $T(\Omega) = T_0 + WT_1(\Omega) + W^2T_2(\Omega) + \cdots$, and is required up to second order. As in the small island case, the calculation of T_1 (and hence J_1) only requires a knowledge of the lowest order flux function $\Omega_0(X, \xi)$. Then, the Ampere equation,

$$\frac{\partial^2\Omega_1}{\partial X^2} = \begin{cases} a\sigma\left[\frac{8}{\sqrt{m}} - 2\pi\int_0^m \frac{dm'}{(m')^{3/2}}\left(\frac{1}{E(m')} - \frac{2}{\pi}\right)\right] \\ 0, \end{cases} \qquad (10)$$

valid, respectively, outside and inside the separatrix, can be integrated to determine the asymmetric correction, Ω_1, to the flux. In Eq. (10), $E(m')$ is the elliptic integral of the second kind. Finally, this permits the calculation of T_2 and J_2 and evaluation of the quantity $M_{\rm in}$. Equating this quantity to $M_{\rm out}$, Eq. (7), one obtains

$$\Delta' = (0.8a^2 - 0.27b)W, \qquad (11)$$

determining the island amplitude when $W \gg W_c$. It is noteworthy that the parameter A, representing the global asymmetry of the tearing mode structure, and the $\ln(W)$ term, do not appear in the large island limit. The result (11) is asymptotically correct in the limit $W_c/W \to 0$, but if finite W_c/W effects were retained, terms involving $\ln(W)$ and A, having their origin in a boundary layer at the separatrix [12], would reappear although multiplied by a factor of order $O(W_c^2/W^2)$ compared with the corresponding terms in Eq. (8).

Finally, we summarize the cylindrical results. There are two important differences between slab and cylindrical geometries. First, the helical and z components of the current density are not the same in cylindrical geometry, but differ by a constant, $J_* = J - 2k_zB_0/m$, with m the poloidal mode number; second, $J_* = -\nabla_\perp^2\Psi_*$ now involves both first and second order radial derivatives of Ψ_* with respect to r. As a consequence, the parameters a and b appearing in the small island solution (10) are changed into $a_* = LJ'_{\rm eq}/J_{\rm eq*} = (1 - 2/s)a$ and $b_* = L^2J''_{\rm eq}/J_{\rm eq*} = (1 - 2/s)b$, where $s = (r/q)(dq/dr)$ is the magnetic shear parameter and $q = 2\pi rB_0/LB_{\theta\rm eq}(r)$ is the magnetic winding index, evaluated at the reconnecting radius $r/L = r_s$. Furthermore, extra terms appear in the island saturation equations, which now read

$$\Delta' = 0.41W\left\{a_*^2\left[\ln\left(\frac{1}{W}\right) + 4.85 - \frac{0.68}{2 - s}\right] - \frac{Aa_*}{2} - b_*\right\} - 0.18(a_*/r_s)W, \qquad (12)$$

$$\Delta' = W[0.8a_*^2 - 0.27b_* - 0.09(a_*/r_s)]. \qquad (13)$$

Eqs. (8) and (11) in slab geometry, and their equivalents in cylindrical geometry, Eqs. (12) and (13), determine the size of the saturated magnetic island, which will develop when the original 1D equilibrium evolves to a state where

$\Delta' > 0$. They are rigorous asymptotic results in their respective limits of validity: $W \ll W_c \ll 1$, for Eqs. (8) and (12), and $W_c \ll W \ll 1$ for Eqs. (11) and (13). In the small island cylindrical result, Eq. (12), we note that the parameters A, a_*, and b_* can take either sign. However the combination $\alpha = 0.41\{[\ln(1/W) + 4.85 - \frac{0.68}{2-s}]a_*^2 - 0.5Aa_* - b_* - 0.44a_*\}$ is strongly positive for a typical current density profile in a tokamak. Taking the peaked profile, $J = J_0(1 - r^2)^2$ (with $r = 1$ the plasma edge) as an example, and taking, for simplicity, $\ln(1/W) = 4$, we find $10 < \alpha(r_s) < 55$ for the $m/n = 2/1$ tearing mode, as the location of the resonant surface r_s is varied. The higher values of α are attained near the plasma center. For this current density profile the 2/1 mode is unstable (with conducting wall boundary conditions) when $r_s \lesssim 0.88$, at which point $\alpha \sim 16$.

Figure 1 shows the saturated island widths as a function of Δ' comparing the small and large island solutions. The solution by Pletzer and Perkins [11] is also shown (Thyagaraja's solution [7] gives a similar curve but its shape depends on the chosen value of the normalization length, L). Note that the dashed curve indicates the existence of a tangent bifurcation with $W < 1$ (see also Ref. [7]) while in our results the tangent bifurcation has disappeared. An interesting conclusion is that, when the small and large island limits are interpolated and included in the time-dependent Rutherford equation, the island width increases considerably as W exceeds W_c, of which a realistic value is indicated in the figure.

In terms of practical relevance of the present analysis and its applicability to realistic tokamak plasmas, the main limitations are the neglect of diamagnetic, ion Larmor radius (ρ_i) and three-dimensional effects. Diamagnetic and Larmor radius effects are known to be important when the island width is relatively small, i.e., $w \lesssim \rho_i$. Three-dimensional effects are important for relatively large islands, i.e., when islands with different helicities initially localized around different resonance surfaces tend to overlap. On the other hand, the rigorous theoretical framework emerging from this Letter is a possible starting point for further generalizations to more complex physical situation. As far as neoclassical effects are concerned, the new terms in Eq. (12) will compete with the neoclassical bootstrap term [12,16], and for the relatively large values of α noted above, we can expect these terms to have some impact on the NTM seed island calculations and on the overall nonlinear tearing saturation level [17].

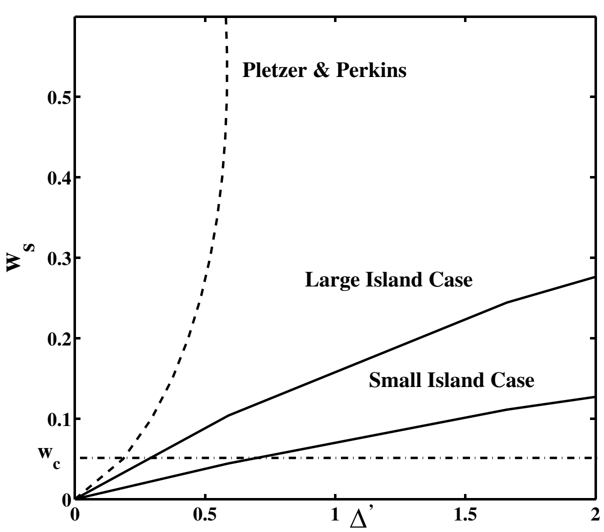

FIG. 1. Saturated island width as a function of Δ'. Comparison between Pletzer *et al.* small [Eq. (12)] and large [Eq. (13)] island results for a typical bell shaped equilibrium current density profile.

In conclusion, by exploiting and extending an asymptotic matching method first described by Thyagaraja [7] we have derived a fully nonlinear solution to the tearing mode saturation problem in both the large and small island limits, in slab and cylinder geometry. For relatively small islands, $W \ll W_c$, our results correct and extend those in previous literature. The case of relatively large islands, where the resistivity profile is modified by the island itself, is considered here for the first time.

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