Low-Frequency Noise Controls On-Off Intermittency of Bifurcating Systems

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A bifurcating system subject to multiplicative noise can display on-off intermittency. Using a canonical example, we investigate the extreme sensitivity of the intermittent behavior to the nature of the noise. Through a perturbative expansion and numerical studies of the probability density function of the unstable mode, we show that intermittency is controlled by the ratio between the departure from onset and the value of the noise spectrum at zero frequency. Reducing the noise spectrum at zero frequency shrinks the intermittency regime drastically. This effect also modifies the distribution of the duration that the system spends in the off phase. Mechanisms and applications to more complex bifurcating systems are discussed.

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Among the possible behaviors of a chaotic system is intermittent behavior. The system remains for long durations in some regular state (say a laminar state or off-phase) and at unpredictable instants begins to explore other states (say on-phase) before returning to the laminar state.

A simple deterministic model for intermittency was proposed by Pomeau and Manneville [1]: a limit cycle is weakly unstable but from time to time a reinjection mechanism forces the system to return close to this limit cycle. A few years later, a new type of intermittency was discovered in coupled dynamical systems [2] and also identified in a system of reaction-diffusion equations [3]. Both systems can be approximately described by the evolution of a weakly linearly unstable mode with a noisy control parameter. This type of intermittency was given the name "on-off intermittency" by Platt, Spiegel, and Tresser [4] who pointed out its genericity when an unstable system is coupled to a system that evolves in an unpredictable manner. Experimentally, on-off intermittency has been identified in various systems including electronic devices, electrohydrodynamic convection in nematics, gas discharge plasmas, and spin-wave instabilities [5].

It is surprising that, despite the genericity of the on-off intermittency mechanism, this effect has not been reported more often. One might expect that any careful experimental investigation of an instability should reveal on-off intermittency when the system is close to the onset of instability, and is hence sensitive to unavoidable experimental noise in the control parameters. This remark is the main motivation for the present work. We show that the amplitude of the noise is not the relevant control parameter of on-off intermittency.

Through an analytical study of a simple stochastic system, we identify the parameter that drives the intermittent behavior and compare our prediction to numerical simulations. We then test our prediction for a chaotic rather than a stochastic system. Then we discuss the sensitivity of the statistics of the laminar phase duration to the parameter that controls the on-off intermittency. Finally, we present applications of this result to complex systems.

One of the simplest systems that can exhibit on-off intermittency is

$$\dot{X} = [a + \zeta(t)]X - X^3, \tag{1}$$

where ζ is a random process with zero mean [2]. In the deterministic regime (no-noise), the variable X undergoes a pitchfork supercritical bifurcation for $a = 0$. The attractor, $X = 0$, is stable for negative a and is unstable for positive a: X tends in the long time limit to one of its two stable attractors $\pm\sqrt{a}$. In the stochastic regime, the noise ζ acts as a modulation of the forcing parameter. Note that if the initial condition verifies $X(t=0) \geq 0$ then $X(t) \geq 0$ for all time. Henceforth we consider only positive initial conditions for X without any loss of generality.

For stationary Gaussian white noise, the probability density function (PDF) of X is derived by solving the associated Fokker-Planck equation [6]. We define the noise intensity by $\langle\zeta(t)\zeta(t')\rangle_s = D\delta(t-t')$, where $\langle\rangle_s$ is the average over realizations of the noise. Equation (1) is then understood in the sense of Stratonovich. If $a \leq 0$, X tends to zero and the stationary PDF is $P(X) = \delta(X)$. For positive a, one obtains

$$P(X) = CX^{(2a/D)-1}e^{-X^2/D}, \tag{2}$$

where C is a normalization constant. For $0 \leq 2a/D < 1$, this PDF diverges at the origin $X = 0$. As pointed out in [2], this divergence is associated with the intermittent behavior of X, as X remains for long durations arbitrarily close to the unstable fixed point $X = 0$. When $2a/D$ is large, intermittency disappears and X fluctuates around its deterministic value, $\sqrt{a}$.

However, for colored noise, a more complex situation is expected. It is tempting to assume that the noise amplitude $\sqrt{\langle\zeta(t)^2\rangle_s}$ controls the on-off regime. This is not the case. We plot in Fig. 1 the solution of Eq. (1) for two different colored noises with the same value of $\langle\zeta^2\rangle_s$. One of the

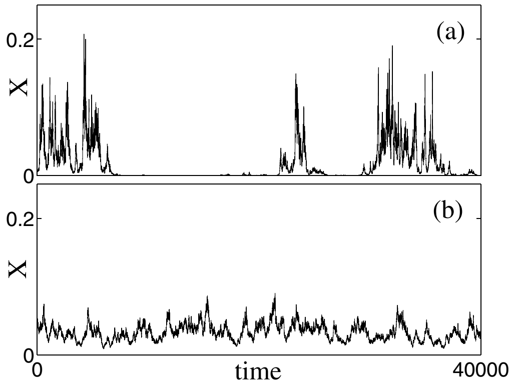

FIG. 1. Temporal traces of the dynamical variable $X(t)$, solution of (1), with $a = 1.25 \times 10^{-3}$, $\alpha^2 = 0.005$ and the noise defined by (6): (a) $\eta = \Omega = 0.25$, i.e., $a/S = 0.3927$; (b) $\eta = \Omega = 2.5$, i.e., $a/S = 3.9270$.

solutions is intermittent but not the other. Hence intermittency is controlled by another parameter of the system.

In order to identify this parameter, we derive an approximate expression of the stationary PDF of X using the cumulant expansion introduced by van Kampen and shown to be valid for small values of $\alpha\tau_c$ where α is the noise amplitude and τ_c its correlation time [7]. In the case under study, two parameters appear in the expansion that are related to the autocorrelation function:

$$S = \int_0^\infty \langle \zeta(0)\zeta(\tau)\rangle_s d\tau, \qquad M = \int_0^\infty \langle \zeta(0)\zeta(\tau)\rangle_s e^{-2a\tau} d\tau. \tag{3}$$

For a Gaussian white noise, the expansion is exact and leads to the stationary PDF given by Eq. (2). In the following we consider the generic case where S and $M - S$ are nonzero. The stationary PDF is

$$P(X) = CX^{a/S-1}\left|1 + \frac{(M-S)X^2}{Sa}\right|^{-[1+aM/2S(M-S)]}, \tag{4}$$

where C is a normalization constant. The behavior of the PDF for X close to zero is proportional to $|X|^{a/S-1}$. It diverges for $X = 0$ so that on-off intermittency occurs if

$$0 < \frac{a}{S} < 1. \tag{5}$$

This is consistent with Fig. 1 since $a/S = 0.3927$ for the intermittent signal and $a/S = 3.927$ for the other one. The Wiener-Khintchin theorem states that the integral of the correlation function, $2S$, is equal to the spectrum of the noise at zero frequency. Thus, another interpretation of the criterion (5) is that on-off intermittency is present when the departure from the deterministic onset is smaller than half the value of the noise spectrum at zero frequency. This interpretation also holds when the noise is white and Gaussian since $S = D/2$. For white noise, the spectrum has the same value, D, for all the frequencies. Consequently, the analytical result (2) does not identify which part of the spectrum controls the intermittency.

In order to check the validity of this result, we solve Eq. (1) numerically, using a stationary Gaussian correlated noise $\zeta(t)$ with autocorrelation function [8]:

$$\langle \zeta(t)\zeta(t+\tau)\rangle_s = \alpha^2\left[\cos(2\pi\Omega\tau) + \frac{\eta}{\Omega}\sin(2\pi\Omega|\tau|)\right] \times \exp(-2\pi\eta|\tau|). \tag{6}$$

The noise variance is α^2 and its correlation time $\tau_c = (2\pi\eta)^{-1}$. This provides $S = \alpha^2\eta/[\pi(\eta^2 + \Omega^2)]$ and $M = \alpha^2(\eta + a/\pi)/\{\pi[(\eta + a/\pi)^2 + \Omega^2]\}$. Therefore, by changing η and Ω we can tune a/S and $\alpha\tau_c$ independently. Gaussian white noise is recovered in the limit $\eta \to \infty$ with $\alpha^2/\eta = D$. Figure 1 presents time series of X and Fig. 2 the corresponding PDFs. We also compare the predicted criterion for appearance of intermittency (5) with the numerical results. To wit, we draw in Fig. 3 a phase diagram in the (S, a) plane using noises with different S and M. We calculate X_{mp} the most probable value of X. For $a > 0$, the solution $X = 0$ is unstable. The system is intermittent if $X_{\mathrm{mp}} = 0$, and nonintermittent if $X_{\mathrm{mp}} \neq 0$. In these figures and for all tested parameter values for which $\alpha\tau_c$ is small, there is a very good agreement between the numerical results and the predictions (4) and (5).

Up to now, we have only dealt with fluctuating parameters that are random processes. It is tempting to test the prediction (4) and (5) with a deterministic but chaotic fluctuating parameter. Thus, we study Eq. (1) when ζ is obtained from the chaotic solution of the Lorenz system [9]; i.e., we solve

$$\dot{U} = -\sigma(U - Y), \qquad \dot{Y} = rU - Y - UZ, \qquad \dot{Z} = UY - bZ, \tag{7}$$

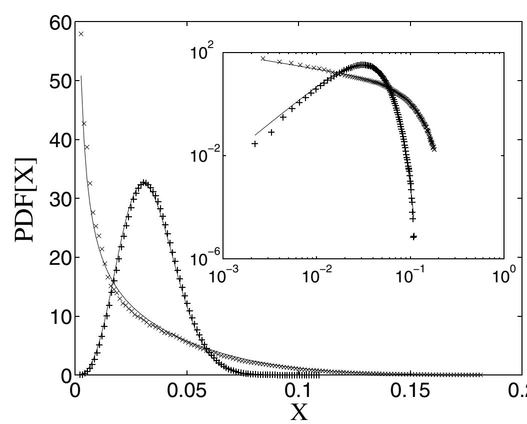

FIG. 2. PDF of the solutions of (1) with the noise (6). The symbols (×) and (+) correspond, respectively, to the parameters used in Figs. 1(a) and 1(b). The full lines are the corresponding predictions given by (4). Same figure in inset using log-log scale.

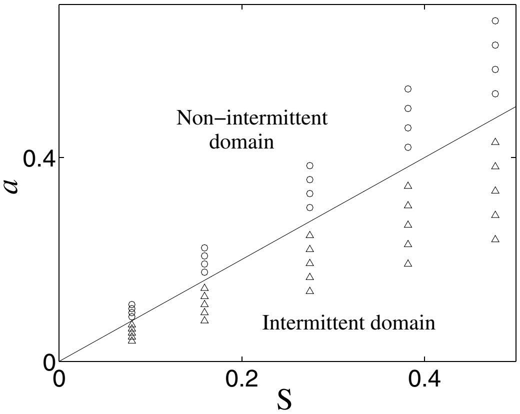

FIG. 3. Behavior of the solution of (1) for noises defined in (6) and associated with various values of S and M. ($\triangle$): intermittent behavior, ($\bigcirc$): nonintermittent behavior. The full line is the transition curve predicted by (5).

and define ζ by

$$\zeta = \alpha \frac{\mu \dot{U}_n + (1-\mu)U_n}{\langle(\mu \dot{U}_n + (1-\mu)U_n)^2\rangle^{1/2}}, \tag{8}$$

$U_n = \frac{U - \langle U\rangle}{\sqrt{\langle(U-\langle U\rangle)^2\rangle}}$, and $\dot{U}_n = \frac{\dot{U} - \langle \dot{U}\rangle}{\sqrt{\langle(\dot{U}-\langle \dot{U}\rangle)^2\rangle}}$. Averages are now understood as long time averages and $\sqrt{\langle\zeta^2\rangle} = \alpha$. The parameter μ is tuned between zero and one to change the amplitude of the noise spectrum at zero frequency. Since $\dot{U}$ is the derivative of U, the value of its spectrum at low frequencies is smaller than that of U. Increasing μ increases the weight of $\dot{U}$ and thus decreases the noise spectrum at low frequencies (accordingly the value of S).

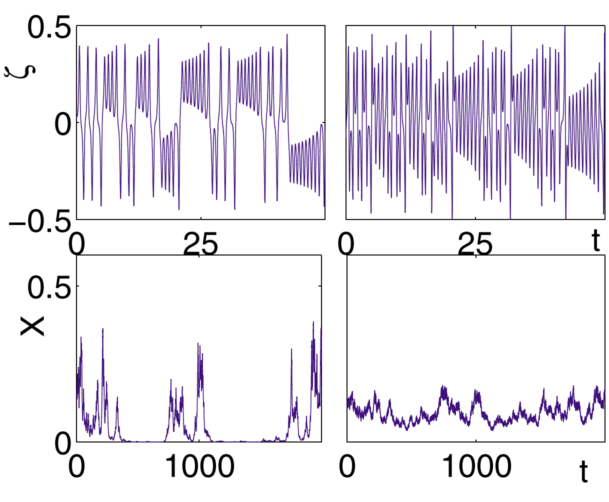

FIG. 4 (color online). Bottom panels: solutions of (1) with ζ obtained from the Lorenz system (8). The chaotic functions ζ are displayed in the top panels. The parameters are $a = 0.01$, $\sqrt{\langle\zeta^2\rangle} = 0.2$, and $\mu = 0$ (left panels), $\mu = 0.8$ (right panels). Note the difference in the time scales t between the top and bottom panels.

The Eqs. (1) and (7) are then solved numerically for $r = 25$, $\sigma = 10$, and $b = 8/3$. The solution of Eq. (7) is chaotic and we plot examples of time series of ζ and X in Fig. 4. As expected, on-off intermittency disappears when μ increases; i.e., S decreases. For the intermittent signal we have $a/S \simeq 0.332$, but $a/S \simeq 5.64$ for the nonintermittent one. The numerical estimates of the PDFs of X are compared to the fit given by (4) (in Fig. 5). Again, for small values of the noise amplitude, the agreement between the prediction and the numerical results is very good.

Interesting results can also be obtained for the duration of the laminar phases τ. We define a laminar phase as follows: $X(t_0) = \epsilon$, $X(t) < \epsilon$ for $t_0 < t < t_0 + \tau$ and $X(t_0 + \tau) = \epsilon$, ϵ being an arbitrary threshold below which the system is considered to be in the laminar state. Close to the onset of on-off intermittency, the probability $P(\tau)$ of the duration τ of the laminar phase satisfies $P(\tau) \propto \tau^{-3/2}$ [10]. For large values of τ, a cutoff in the power law appears at finite departure from onset [11]. We have checked numerically that for a colored noise, the PDF is indeed proportional to $\tau^{-3/2}$ with a cutoff for high τ. The position of the cutoff increases when S increases (a being constant). This is consistent again with our interpretation of the role of the noise spectrum at zero frequency: the smaller a/S is, the longer the system remains in the laminar state and the more intermittent the signal appears.

Our interpretation of the phenomenon is as follows. On-off intermittency occurs because of a competition between the noise and a systematic drift driven by the distance from onset. More precisely, as pointed out in [2], when X is close to the unstable manifold $X = 0$, the evolution of $Y = \log X$ is given by $\dot{Y} = a + \zeta(t)$. For positive a, $\dot{Y}$ is positive on average but events in which Y remains smaller than its initial value are possible provided $I = \int_0^T \zeta(t)dt/T$ remains smaller than $-a$ for a long duration. In the long time limit, the main contribution to the integral I is due to

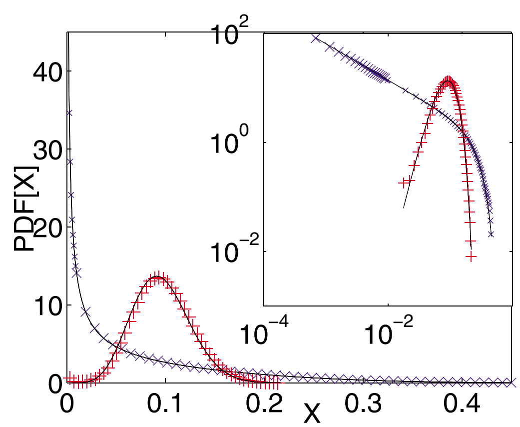

FIG. 5 (color online). PDF of the solutions of (1) with ζ obtained from the Lorenz system (8). The parameters are the same as in Fig. 4 with ($\times$) $\mu = 0$, (+) $\mu = 0.8$. The inset is the same figure with log-log scales. The full lines are the corresponding predictions (4).

the zero frequency component of the noise. If this component is reduced, then occurrences of the inequality $I \leq -a$ are less probable and intermittency tends to disappear. Note that our analytical calculations are based on perturbative expansions and are valid for small values of the product of the noise amplitude with its correlation time. However, even for a finite amplitude of noise, we have verified that when the low frequencies are filtered out, intermittency disappears.

We expect that the role of the zero frequency component of the noise is generic and also pertinent for systems more complex than the one presented here. For instance, Lücke and Schanck studied a system similar to Eq. (1) with inertia taken into account. Through a perturbative expansion close to the deterministic solution, they calculated a noise-induced postponement of the onset of instability and a modification in the amplitude of the unstable mode. As they pointed out later, their expansion is not correct when the noise spectrum does not vanish at zero frequency [12]. Recent calculations for the same system have shown that for a small departure from onset and for an Ornstein-Uhlenbeck or a white noise, the PDF of the unstable mode diverges close to zero [13]. In the light of our study, both the divergence of the PDF and the failure of the perturbative expansion are related to the same physical effect: on-off intermittency when the noise spectrum at zero frequency is nonzero.

Our analysis explains why many experimental investigations on the effect of a multiplicative noise on an instability do not display on-off intermittency. If the noise is high-pass filtered, as often required for experimental reasons, then the regime of intermittent behavior disappears. This is the case, for instance, in [14]: a ferrofluidic layer undergoes the Rosensweig instability and peaks appear at the surface. The layer is then subject to a multiplicative noise through random vertical shaking. Close to the deterministic onset, the unstable mode submitted to a colored noise does not display intermittency.

In dynamo theory, the magnetic field is forced by the flow of an electrically conducting fluid. The velocity of the flow appears as a multiplicative term in the equation for the magnetic field. If the flow topology is complex enough and the velocity is large, a magnetic field is generated by dynamo instability. The flow is in general turbulent at dynamo onset so that the velocity fluctuates. We infer that the intermittent behaviors as seen numerically by Sweet *et al.* [15] are related to the presence of very low frequencies in the spectrum of the velocity field. The same features also occur in simple models of dynamos subject to white noise [16]. On the contrary, experimental realizations of dynamos driven by constrained flows did not display intermittency [17]. These flows, though turbulent, are probably too constrained to display velocity fluctuations with large enough amplitude of the spectrum at low frequencies.

This work could be generalized to study a parametric instability with a time-dependent forcing. For a harmonic forcing subject to frequency or amplitude noise, intermittent behaviors have been reported [18]. In these cases, the relevant component of the noise that controls on-off intermittency still remains to be identified.

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