Nonsingular Black Holes and Degrees of Freedom in Quantum Gravity

Martin Bojowald*

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany

(Received 19 April 2005; published 2 August 2005)

Spherically symmetric space-times provide many examples for interesting black hole solutions, which classically are all singular. Following a general program, spacelike singularities in spherically symmetric quantum geometry, as well as other inhomogeneous models, are shown to be absent. Moreover, one sees how the classical reduction from infinitely many kinematical degrees of freedom to only one physical one, the mass, can arise, where aspects of quantum cosmology such as the problem of initial conditions play a role.

DOI: 10.1103/PhysRevLett.95.061301 PACS numbers: 04.60.Pp, 04.70.Dy

One of the main issues to be addressed by quantum gravity is the singularity problem of general relativity. While the classical theory is very successful in describing space-time on scales that can be probed today, it is incomplete because it predicts the generic presence of singularities: boundaries of space-time that can be reached by observers in a finite amount of time, but at which point the theory becomes inapplicable. Usually, curvature or energy densities and tidal forces diverge there, implying unphysical conditions.

One explanation is that the picture of a smooth space-time underlying the classical theory is appropriate only at large scales, while at small scales the structure is discrete. This has, indeed, been substantiated by looking at cosmological models in loop quantum gravity [1], where the discrete quantum geometry has been shown to remove singularities [2]. Even at the classical level there are indications for a breakdown of the smooth picture at small scales: The Belinskii-Khalatnikov-Lifschitz (BKL) scenario [3] provides a scheme for the general approach to a classical singularity by considering dominant contributions to the field equations in this limit. It turns out that only terms with time derivatives remain such that spatial points decouple and their geometries can be described by the most general homogeneous model. This Bianchi type IX model is classically chaotic [4], and so geometries in different points are completely unrelated, implying a complicated classical singularity with structure at arbitrarily small scales.

Since the removal of singularities in loop quantum cosmology applies for all homogeneous models [5,6], in particular, the Bianchi IX model, one can combine this result with the BKL picture and expect that all singularities are removed by quantum geometry. However, the BKL picture has not been proven classically, and the above argument would require it to hold even in quantum gravity. In particular, the latter point is questionable because, for one, already the approach to a single Bianchi IX singularity [4] in quantum cosmology is modified, removing the classical chaos [7,8]. It is thus necessary to study inhomogeneous models in loop quantum gravity and look at the singularity issue without assuming homogeneity. If this is possible, one can also test the validity of the BKL picture in the quantum context. This is what we do here in the case of spherical symmetry, which is not only the simplest inhomogeneous situation but also allows interpretations for black holes. The same methods apply to other models that do have local degrees of freedom, providing the first demonstration of the absence of singularities in inhomogeneous quantum gravity.

Singularities.—The main problem caused by a singularity is the fact that it presents a boundary to physical evolution. In order to see whether it persists in quantum gravity, then, the following steps have to be performed. This has to be done in a manner that is independent of coordinate or other gauge choices, and only potential simplifications resulting from the symmetry reduction should be used. One first has to locate classical singularities on the phase space of physical fields, the spatial metric q_{ab}, and extrinsic curvature related to $\dot{q}_{ab}$. Conditions to specify the *singular part of phase space* must be chosen such that any solution to the theory, which is a trajectory on phase space, intersects this singular part exactly when it develops a singularity. The solution space is, in general, quite complicated to study, but one can select a variable T on phase space that is transversal to the singular part, a *local internal time* rather than coordinate time, i.e., which fulfills $T = 0$ in a neighborhood about zero exactly at the singular part. Finally, one needs to write down the quantum evolution of geometry in the local internal time and check whether or not it stops at $T = 0$. If one can find a T such that the quantum evolution does not stop anywhere, the quantum system is nonsingular. This is the analog of the classical notion of space-time completeness.

We illustrate this scheme with isotropic cosmology where the phase space is two dimensional with the scale factor a (the spatial radius of the universe) and its time derivative. Singularities occur only if the scale factor vanishes such that $a = 0$ specifies the singular part. An obvious local internal time (which in this case is global) is given by $T = a$, or with a slight modification the triad variable p with $|p| = a^2$ and $\mathrm{sgn}p$ being the orientation of space. Using this variable makes no difference classically, but is important in quantum geometry where triads are basic variables. At the quantum level one can then first note that operators for p^{-1} are finite [9], indicating already that curvatures and energy densities do not diverge, and most importantly that the quantum evolution is given by a difference equation for the wave function in p, which *does not stop at* $p = 0$ [2]. Thus, there is no singularity in isotropic loop quantum cosmology.

Spherical symmetry.—The case of interest here is spherical symmetry, where the kinematical phase space on which we have to locate singularities is infinite dimensional and spanned by the metric components in $ds^2 = q_x(x)dx^2 + q_\Omega(x)d\Omega^2$ (in polar coordinates) and their time derivatives. As an example, we can look at the Schwarzschild solution for a black hole of mass M, $q_x = (1 - 2M/x)^{-1}$, $q_\Omega = x^2$. The singularity is reached for $x = 0$, at which point both metric coefficients are zero. The question then arises which one, or both, of them must be zero as a condition for a singularity. It turns out that q_Ω is zero only at the singularity, while q_x can also become zero elsewhere, i.e., at the horizon $x = 2M$, when one chooses a different gauge (e.g., with homogeneous coordinates in the interior). This point illustrates why gauge independence is essential in answering the singularity problem: even the very first step, finding where singularities would develop, depends on it. In fact, in this case we can choose our coordinates x and t at will, which affects the form of q_x and points where it can be zero. In spherical symmetry, however, the fact that $q_\Omega = 0$ at the singularity is unaffected (even though, of course, q_Ω can change as a function of x when we change coordinates).

We can now consider a spatial slice that locally, around a point x_0, approaches the classical singularity such that $q_\Omega(x_0) \to 0$. The above discussion shows that $T = q_\Omega(x_0)$ is a good local internal time, which completes setting up the problem from the classical side. It now remains to formulate the quantum evolution in local internal time and to check if it stops at $T = 0$.

Quantum geometry.—Again we first transform to triad variables $|E^x| = q_\Omega$ and $E^\varphi = \sqrt{q_x q_\Omega}$, which become basic operators in quantum geometry. The (local) orientation of space around a point x_0 is now given by $\mathrm{sgn}E^x(x_0)$ where E^x unlike E^φ can take both signs. Moreover, the discussion in metric variables shows that $T = E^x(x_0)$ is our local internal time such that the situation, so far, is analogous to that in the isotropic case: triad variables lead to a local internal time, which takes values at both sides of the classical singularity, $T = 0$ defining a manifold in superspace rather than at the boundary. It is important to note that the introduction of triad variables was seen as a necessary step toward a background independent quantization. Now it turns out that this also changes the singularity structure on phase space in a way that was important for removing cosmological singularities. Nevertheless, even though the singularity is now located in superspace, the classical evolution still stops there and is not able to connect from positive to negative T. This still has to be checked by the quantum evolution, the most crucial point.

Quantum evolution follows from the Hamiltonian constraint operator acting on states in the form of a lattice model with basis [10]

$$|\vec{k},\vec{\mu}\rangle := \cdots \underset{\cdots}{\bullet} \overset{k_n}{\longrightarrow} \underset{\mu_n}{\bullet} \overset{k_{n+1}}{\longrightarrow} \underset{\cdots}{\bullet} \cdots$$

where the integer labels k_e on edges are eigenvalues of the operator $\hat{E}^x$ and the positive real labels $\mu(v)$ at vertices are those of $\hat{E}^\varphi$. Positions of vertices do not refer to a background space, and the lattice model represents the continuum theory. The constraint then acts by [11]

$$\hat{H}[N] \overset{k_-}{\text{—}}\bullet\overset{k_+}{\text{—}} = \sum_v N(v)\Big(\hat{C}_0(k) \overset{k_-}{\text{—}}\bullet\overset{k_+}{\text{—}} + \hat{C}_{R+}(k) \overset{k_-}{\text{—}}\bullet\overset{k_++2}{\text{—}} + \hat{C}_{R-}(k) \overset{k_-}{\text{—}}\bullet\overset{k_+-2}{\text{—}}$$
$$+ \hat{C}_{L+}(k) \overset{k_-+2}{\text{—}}\bullet\overset{k_+}{\text{—}} + \hat{C}_{L-}(k) \overset{k_--2}{\text{—}}\bullet\overset{k_+}{\text{—}} + \cdots\Big)$$

summing over all vertices of the lattice, the dots indicating further terms such as a matter Hamiltonian whose detailed form is not important here. The known coefficients $\hat{C}_I(k) = C_I(k)\hat{C}_I$ consist of functions $C_I(k)$ of the edge labels and operators $\hat{C}_I$ acting only on the dependence on vertex labels μ. A general state is now a superposition $|\psi\rangle = \sum_{\vec{k},\vec{\mu}} \psi(\vec{k},\vec{\mu})|\vec{k},\vec{\mu}\rangle$ whose coefficients $\psi(\vec{k},\vec{\mu})$ define the state in the triad representation. The constraint $\hat{H}[N]|\psi\rangle = 0$ has to hold true for all functions N with independent values $N(v)$, giving one equation for each vertex that in the triad representation takes the form

$$\hat{C}_0(k)\psi(k_-,k_+) + \hat{C}_{R+}(k)\psi(k_-,k_+-2) + \hat{C}_{R-}(k)\psi(k_-,k_++2) + \hat{C}_{L+}(k)\psi(k_--2,k_+) + \hat{C}_{L-}(k)\psi(k_-+2,k_+) + \cdots = 0$$

of a difference equation, where we have suppressed the vertex labels on which the $\hat{C}_I$ act and unchanged k.

We now solve this set of equations with initial and boundary values for the wave function. To define a solution scheme we proceed iteratively from vertex to vertex, starting at one side ∂ of the lattice. We assume that the boundary values for all μ_∂ and $k_+(\partial) =: k_-$ of the wave function as well as values for large positive $k_e = k_0$ and $k_0 - 1$ at all edges e are given, which means that we have specified the initial situation, e.g., by a semiclassical state specifying the initial slice far away from the singularity. The equation can then be solved for $\hat{C}_{R+}\psi(k_-,k_+-2)$ in terms of values of the wave function specified by the initial conditions. This brings us one step further because we now have information about the wave function at $k_+ - 2$ for a smaller edge label (our local internal time) evolving toward the classical singularity.

Next, we have to know how to find ψ from its image under $\hat{C}_{R+}$. This can be done by specifying conditions for the wave function at small μ (which is not in the singular part of minisuperspace but represents an ordinary boundary) and happens in exactly the same way as in homogeneous models [5]. Before continuing, we notice that this indicates the presence of aspects of the BKL picture in quantum gravity. However, we still have to try to evolve through the classical singularity, i.e., $k_e = 0$, which will be the main test. One crucial difference to cosmological models is that the coefficients $\hat{C}_I(k)$ are not only functions of the local internal time, k_+, studied in the iteration but also of neighboring labels such as k_-, which do not take part in this difference equation but the dependence on which has been determined in iteration steps for previous vertices. This is clearly a new feature coming from the inhomogeneous context, and it has a bearing on the singularity issue.

Singularities are removed if the difference equation determines the wave function everywhere on minisuperspace once initial and boundary conditions have been chosen away from classical singularities. The simplest realization is by a difference equation with nonzero coefficients everywhere. However, this is not automatically the case with an equation coming from a general construction of the Hamiltonian constraint, and so has to be checked explicitly. Here, it turns out [11] that a symmetric constraint, indeed, leads to nonzero functions $C_I(k)$, which then will not pose a problem to the evolution. All values of the wave function, at positive as well as negative k, are determined uniquely by the difference equations and chosen initial and boundary values. The evolution thus continues through the classical singularity at zero k: *there is no quantum singularity.* Other quantization choices can lead to quantum singularities, providing selection criteria to formulate the quantum theory with implications also for the full framework.

Consequences.—We have shown that the same mechanism as in homogeneous models contributes to the removal of spherically symmetric classical singularities. Key features are that densitized triads as basic variables in quantum geometry provide us with a local internal time taking values at *two sides of the classical singularity,* combined with a quantum evolution that *connects both sides.* No new ingredients are necessary for inhomogeneous singularities, only an application of the general scheme to the new and more complicated situation.

As in cosmological models, the argument applies *only to spacelike singularities* such as the Schwarzschild one. The reason is that we evolve a spatial slice toward the classical singularity and test whether it will stop. A timelike or null singularity would require a different mechanism that is not known at present. Thus, cases like negative mass solutions seem to remain singular, which is a welcome property helping to rule out unwanted solutions leading to instability [12].

This scenario and its form of difference equations do not only apply to vacuum black holes but also to spherically symmetric matter systems. In such a case, there would be new labels for matter fields, and a contribution to the constraint from the matter Hamiltonian. As this does not change the structure of the difference equation, the same conclusions apply. Moreover, models for Einstein-Rosen waves have a similar structure just with a new vertex label. Also in this case, with or without matter fields, the analysis goes through such that the absence of singularities can be demonstrated even in situations with local gravitational degrees of freedom.

There are differences between homogeneous models and these inhomogeneous cases, and the inhomogeneous analysis is much more nontrivial. In homogeneous models there are several ambiguities in the constraint operator, and several choices lead to nonsingular evolution. In more complicated situations such as those studied here, not all options remain available. In particular, we had to use a symmetric ordering of the constraint in order to have nonvanishing coefficients of the difference equation. In homogeneous models one can also work with a version whose coefficients vanish right at the singularity. The evolution then still continues since the value at the classical singularity simply decouples and does not play a role for the evolution. Instead, one can use the behavior to find dynamical initial conditions [13]. This is also possible here for evolution in local internal time, but then the decoupled value at $k_- = 0$ is not determined and in general is needed for the wave function at other values of k_+. The inhomogeneous evolution would thus break down, and this choice of constraint is ruled out.

There is a difference in the constraint operator we used compared to a common expression in the full theory [14]. This issue is visible only in inhomogeneous models, and consists in whether or not the constraint creates new edges and vertices, or just changes labels of existing ones. We chose the second possibility, which has already been considered as a modification in the full theory [15]. There, it can better explain the presence of correlations at an intuitive level, but makes checking anomaly freedom more complicated. The main problem of an anomalous quantization would be that too many states could be removed when imposing the constraints, leaving not enough physical solutions. This issue can be checked here with the constraint we used. If there is no matter field present, we expect just one physical degree of freedom, the Schwarzschild mass M. In our solution scheme we started with a boundary state ψ_∂ corresponding to this degree of freedom, and with this state being free it is already clear that we do not lose too many states. It is even possible to check whether or not the number of independent physical solutions is correct, i.e., not too large either. In the iteration we solve one difference equation for ψ at each vertex, such that any freedom here would provide new quantum degrees of freedom. Since the difference equation for ψ has the same form as that in homogeneous loop quantum cosmology, the *number of quantum degrees of freedom is formally related to the initial value problem of quantum cosmology.* A possible physical meaning is to be checked in explicit examples.

In the isotropic case there are, indeed, dynamical initial conditions following from the dynamical law [13,16] which, if realized in our context, would imply that solutions for ψ are unique and the mass is the only quantum degree of freedom. However, these conditions rely on the fact that leading coefficients of the difference equation can vanish, which we have ruled out for inhomogeneous models. Moreover, the uniqueness of a quantum cosmological wave function depends on the preclassicality condition of [13]. Other mechanisms to select unique cosmological solutions are thus needed, such as from observables or the physical inner product [17]. This issue is quite complicated for difference equations, in particular, in anisotropic models [18]. Nonetheless, a simple counting of free variables supports the connection to initial conditions: The vacuum spherically symmetric case has difference equations in three independent variables, an edge label k and two neighboring vertex labels μ. Homogeneous loop quantum cosmology gives rise to an equation of similar structure and also three variables, so if we assume that there is a mechanism for a unique solution it will also apply to black holes of a given mass. Adding matter fields (or more gravitational freedom as in Einstein-Rosen) increases the number of independent variables to five in inhomogeneous models (*two* new vertex labels) as opposed to four in homogeneous matter models. The type of difference equations thus agrees in homogeneous and inhomogeneous models in vacuum, but not when local degrees of freedom are present.

Thus, the structure of the Hamiltonian constraint equation from loop quantum gravity can potentially provide explanations for issues as diverse as the singularity problem in cosmology and black hole physics, initial conditions in quantum cosmology, the semiclassical limit, and the issue of quantum degrees of freedom. We emphasize that many of these connections still have to be checked in generality. Still, such connections between seemingly unrelated issues in quantum gravity can be seen as support for the internal consistency of the whole theory and, hopefully, provide guidance for future developments.

We can finally come back to the approach to a classical singularity and the BKL picture. Our results here do not rely on an extension of the BKL picture to the quantum situation. First of all, the situation is conceptually different because evolution is now studied for a wave function in local internal time T, rather than the spatial metric in coordinate time. Nevertheless, at first sight a similar picture arises here from the quantum equation: as used in the previous arguments, the equations can be reduced to ordinary difference equations in T, where neighboring edges just contribute via an inhomogeneity of the difference equation. The inhomogeneous situation, however, does play an important role right at the classical singularity where some versions that would be allowed in homogeneous models are ruled out.

Given that the techniques necessary for the quantum theory are similar to lattice models, it is easy to implement them in numerical quantum gravity. This opens the door to numerical investigations of many problems that are still actively pursued in classical gravity [4], such as the approach to classical singularities and the issue of gravitational collapse and naked singularities. This requires studying horizons in addition to classical singularities, which can also be done at the quantum level [19]. As we have seen, there are many nontrivial quantum effects that play together in just the right way to ensure the absence of singularities, which has prospects for other effects in the physics of black holes [20].

*Electronic address: mabo@aei.mpg.de

[1] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004); C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, U.K., 2004); T. Thiemann, gr-qc/0110034.

[2] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001).

[3] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifschitz, Adv. Phys. **13**, 639 (1982).

[4] B. Berger, Living Rev. Relativity **5**, 1 (2002).

[5] M. Bojowald, Classical Quantum Gravity **20**, 2595 (2003).

[6] M. Bojowald, G. Date, and K. Vandersloot, Classical Quantum Gravity **21**, 1253 (2004).

[7] M. Bojowald and G. Date, Phys. Rev. Lett. **92**, 071302 (2004).

[8] M. Bojowald, G. Date, and G. M. Hossain, Classical Quantum Gravity **21**, 3541 (2004).

[9] M. Bojowald, Phys. Rev. D **64**, 084018 (2001).

[10] M. Bojowald, Classical Quantum Gravity **21**, 3733 (2004).

[11] M. Bojowald and R. Swiderski (to be published).

[12] G. T. Horowitz and R. C. Myers, Gen. Relativ. Gravit. **27**, 915 (1995).

[13] M. Bojowald, Phys. Rev. Lett. **87**, 121301 (2001).

[14] T. Thiemann, Classical Quantum Gravity **15**, 839 (1998).

[15] L. Smolin, gr-qc/9609034.

[16] M. Bojowald, Gen. Relativ. Gravit. **35**, 1877 (2003).

[17] K. Noui, A. Perez, and K. Vandersloot, Phys. Rev. D **71**, 044025 (2005).

[18] D. Cartin and G. Khanna, Phys. Rev. Lett. **94**, 111302 (2005).

[19] M. Bojowald and R. Swiderski, Phys. Rev. D **71**, 081501(R) (2005); V. Husain and O. Winkler, gr-qc/0412039.

[20] A. Ashtekar and M. Bojowald, gr-qc/0504029.