Dynamic Supersymmetries of Differential Equations with Applications to Nuclear Spectroscopy

F. Iachello

Center for Theoretical Physics, Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06520-8120, USA

(Received 25 March 2005; published 28 July 2005)

Dynamic supersymmetries of differential equations are defined. The case of a liquid drop with quadrupole deformation coupled to a particle with $j = 3/2$ is shown as an example of a situation where the dynamic supersymmetry $OSp(5/4)$ may occur. A special solution, called $E(5/4)$, of interest in the spectroscopy of odd-even nuclei in the transitional region between spherical and gamma unstable is explicitly worked out.

DOI: 10.1103/PhysRevLett.95.052503 PACS numbers: 21.60.Fw, 21.10.Re, 21.60.Ev

Dynamic supersymmetries, that is, situations in which the Hamiltonian operator describing a (nonrelativistic) Bose-Fermi system can be written in terms only of invariant operators of a superalgebra g^* and its (graded or not) subalgebras $g^* \supset g'^* \supset g''^* \supset \ldots$, have played an important role in the development of nuclear physics in the last 25 years. Discovered in the early 1980s [1,2], they have been confirmed recently in a series of experiments performed at various laboratories [3]. The major role of these symmetries is that of providing a classification scheme for complex systems [4]. In fact, the more complex the system is, the more useful the symmetries are, as it is practically impossible at the present time to calculate *ab initio* the properties of these systems.

Dynamic supersymmetries have been so far studied algebraically by realizing the superalgebra g^* in terms of creation and annihilation operators for bosons and fermions [4,5]. For other applications of this concept, it is of interest to study dynamic supersymmetries of differential equations, $D\psi = E\psi$, where D is the differential operator representing the Hamiltonian. In this Letter, dynamic supersymmetries of differential equations are defined, and a specific example of current interest in nuclear spectroscopy, where these situations may occur, is discussed.

In order to study supersymmetries of differential equations, the superalgebra g^* needs to be realized in terms of coordinates and momenta. A realization, often used in high energy physics, is in terms of coordinates $\alpha_\mu(\mu = 1, 2, \ldots, n)$ and momenta $\frac{\partial}{\partial \alpha_\mu}$ for bosons, and Grassmann coordinates $\theta_i(i = 1, \ldots, m; m = \text{even})$ and momenta $\frac{\partial}{\partial \theta_i}$ for fermions. A generic Hamiltonian for mixed systems of bosons and fermions is

$$H = H_B\left(\alpha_\mu, \frac{\partial}{\partial \alpha_\mu}\right) + H_F\left(\theta_i, \frac{\partial}{\partial \theta_i}\right) + V_{BF}\left(\alpha_\mu, \frac{\partial}{\partial \alpha_\mu}; \theta_i, \frac{\partial}{\partial \theta_i}\right). \quad (1)$$

Dynamic supersymmetries of the differential equation (1) can then be defined in the usual way, as those situations in which the Hamiltonian H can be written in terms of Casimir invariants of a superalgebra g^* and its (graded or not) subalgebras $g^* \supset g'^* \supset \cdots$.

The use of Grassmann variables in the solution of the differential equation $H\psi = E\psi$ is rather complex. A simpler realization, often used in quantum mechanics, is in terms of coordinates and momenta for the elements of the bosonic algebra and in terms of $m \times m$ matrices, $m =$ even, for the elements of the fermionic algebra (for spin $j = 1/2$ particles, the Pauli matrices). In the application described in the paragraphs below, the latter realization will be used.

A nontrivial example of supersymmetry of differential equations is the case of a liquid drop with quadrupole deformation $\alpha_\mu(\mu = 0, \pm 1, \pm 2)$ coupled to a particle with spin $3/2$. The differential equation for this case can be cast into a simple form by noting that the coordinates and momenta of the drop, $\alpha_\mu, \frac{\partial}{\partial \alpha_\mu}$, transform as the vector representation $(1, 0)$ of the five-dimensional rotation group $SO(5)$, while the four components $m_j = \pm\frac{1}{2}, \pm\frac{3}{2}$ of the spin $3/2$ particle transform as the representation $(1, 0)$ of the symplectic group $Sp(4)$ [6], or conversely as the *spinor* representation $[\frac{1}{2}, \frac{1}{2}]$ of $SO(5)$. [When spinor representations are included, the corresponding group is called $Spin(5)$ [1]. Details of the group theoretic description will be omitted from this Letter and will be given in a subsequent longer publication.] The Hamiltonian for the drop, the particle, and their interaction is $H = H_B + H_F + V_{BF}$. Consider now the case in which the Hamiltonian H_B of the drop is the Bohr Hamiltonian [7] in coordinates $\beta, \gamma, \vartheta_i(i = 1, 2, 3)$, with a γ-independent potential $V(\beta)$. Furthermore, let the Hamiltonian H_F of the particle be a constant (chosen to be zero), and let the interaction V_{BF} between the drop and the particle be a $Spin(5)$ scalar. This interaction is written as $\boldsymbol{\Sigma} \circ \mathbf{L}$, a five-dimensional "spin-orbit" interaction, generalization of the familiar three-dimensional spin-orbit interaction $\sigma \cdot \ell$. (Note, however, the difference between the five-dimensional dot product $\circ$ and the three-dimensional dot product $\cdot$.) The 10 matrices $\Sigma_{\mu\nu}$ are the 4×4 matrices of $Sp(4)$, tabulated in [8], while $L_{\mu\nu} = \alpha_\mu \frac{\partial}{\partial \alpha_\nu} - \alpha_\nu \frac{\partial}{\partial \alpha_\mu}$ are the 10 components of the five-dimensional angular momentum. The coefficient in front of the dot product is taken to be γ independent. The explicit form of H is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_\kappa \frac{Q_\kappa^2}{\sin^2(\gamma - \frac{2}{3}\pi\kappa)}\right] + V(\beta) + g(\beta)\left[2\boldsymbol{\Sigma}\circ\mathbf{L} + \frac{5}{2}\right]. \quad (2)$$

(The constant $\frac{5}{2}$ has been added for convenience.) This equation is the generalization to five dimensions of the familiar equation of a spin 1/2 particle in a central potential and can be solved with standard techniques. In particular, a dynamic Bose-Fermi symmetry of this equation will occur whenever H can be written in terms of invariant Casimir operators. This will happen when the function $g(\beta) = k\frac{\hbar^2}{2B\beta^2}$, where k is an arbitrary constant. Under the circumstances described above, the eigenvalue equation $H\Psi = E\Psi$ can be separated by writing

$$\Psi = F(\beta)\Phi(\gamma, \vartheta_i, \eta). \quad (3)$$

Here η represents generically the coordinates of the spin 3/2 particle. The wave function Φ is obtained by coupling the wave function of the drop φ with that of the particle χ

$$\Phi_{[\tau_1,1/2],\nu_\Delta,J,M_J} = \sum_{\ell,m_\ell}\left\langle \begin{matrix} [\tau,0] & [\frac{1}{2},\frac{1}{2}] & | & [\tau_1,\frac{1}{2}] \\ \ell & \frac{3}{2} & | & J \end{matrix}\right\rangle\left\langle \begin{matrix} \ell & \frac{3}{2} & | & J \\ m_\ell & m_j & | & M_J \end{matrix}\right\rangle \varphi_{[\tau,0],\nu_\Delta,\ell,m_\ell}(\gamma,\vartheta_i)\chi_{[1/2,1/2],0,3/2,m_j}(\eta). \quad (4)$$

The symbols represent Clebsch-Gordan coefficients for the chain $Spin(5) \supset Spin(3) \supset Spin(2)$ tabulated in [4] (see p. 49). The wave functions are labeled by the representations of $Spin(5)$, with the quantum numbers $[\tau_1, \tau_2]$, ν_Δ, J, M_J discussed in [4]. Insertion of Eq. (3) into Eq. (2) gives the equation for $F(\beta)$:

$$\left[-\frac{\hbar^2}{2B}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\hbar^2}{2B}\frac{\Lambda}{\beta^2} + V(\beta)\right]F(\beta) = EF(\beta), \quad (5)$$

with

$$\Lambda = \tau(\tau+3) + k\left[\tau_1(\tau_1+3) + \frac{3}{4} - \tau(\tau+3)\right]. \quad (6)$$

For applications to spectroscopy, it is also important to compute electromagnetic transition rates. Particularly interesting are the matrix elements of the electromagnetic transition operators. These operators can be written in the generic form

$$T = T_B(\alpha_\mu) + T_F(\eta). \quad (7)$$

In the case described here of a drop coupled to a particle with spin 3/2, the quadrupole transition operator has contributions both from the drop, $t\alpha_\mu$, and the single particle, written as $t'Y_{2\mu}$. One notes again that $\alpha_{2\mu}$ transforms as the representation $(1,0)$ of $SO(5)$. Bayman and Silverberg [6] showed years ago that the single particle contribution $Y_{2\mu}$ transforms as the representation $(1,1)$ of $Sp(4)$, or conversely the representation $[1,0]$ of $Spin(5)$. Thus

$$T_\mu^{(2)} = t\beta T_{[1,0],0,2,\mu}(\gamma,\vartheta_i) + t'T_{[1,0],0,2,\mu}(\eta). \quad (8)$$

The tensorial character of $T^{(2)}$ under $Spin(5)$ can be used to perform the calculation of the angular part (γ, θ_i, η). This calculation was already done using algebraic methods [9]. The only remaining part is the β part. This is done by evaluating the integrals

$$I_{\xi,\tau_1;\xi',\tau_1'} = \int_0^\infty \beta F_{\xi,\tau_1}(\beta)F_{\xi',\tau_1'}(\beta)\beta^4 d\beta \quad (9)$$

for the contribution of the drop and a similar integral without the first factor β for the single particle.

The explicit form of the equation in the β variables shows that among the possible dynamic Bose-Fermi symmetries of differential equations there is that in which $V(\beta) = \beta^2$, for, in that case, the eigenvalue problem can be solved in explicit analytic form. This case is of interest in nuclear spectroscopy, but it will not be discussed here. Instead, I note that the method introduced here can be used to provide a solution to another problem of current interest. Recently, a new concept has been introduced, called "critical symmetry" [10]. One attempts to describe, in explicit analytic form, the situation in which a physical system is at the critical value of a (shape) phase transition. The method discussed here can be used to study situations in which a spin 3/2 particle is coupled to a system at the critical value of the phase transition between spherical and γ-unstable shape. At this point, the potential $V(\beta)$ is flat and can be approximated by a five-dimensional square well, $V(\beta) = 0$, for $\beta \le \beta_W$, and $V(\beta) = \infty$, for $\beta > \beta_W$ [10]. The eigenvalue problem for this case, which has Bose-Fermi symmetry $Spin(5)$, can be solved in explicit form. The solutions of Eq. (2) are given in terms of Bessel functions of order $\nu = \sqrt{\Lambda + \frac{9}{4}}$. (For $k = 1$, $\nu = \sqrt{\tau_1(\tau_1+3) + 3}$.) They can be written as $F_{\xi,\tau_1}(\beta) = c_{\xi,\tau_1}\beta^{-3/2}J_\nu(x_{\xi,\tau_1}\beta/\beta_W)$, where c_{ξ,τ_1} is a normalization constant. The boundary condition at $\beta = \beta_W$ determines the eigenvalues to be

$$E = \frac{\hbar^2}{2B}\left(\frac{x_{\xi,\tau_1}}{\beta_W}\right)^2, \quad (10)$$

where x_{ξ,τ_1} is the ξth zero of $J_\nu(z)$. The spectrum when $k = 1$ is given in Fig. 1. It should be noted that, when $k = 1$, energies are given in terms of only an overall scale. The values of the angular momenta contained in each multiplet τ_1 are given by the reduction $Spin(5) \supset Spin(3)$ and are tabulated in [4] (see p. 43).

The calculation of the matrix elements of the transition operators is also straightforward. One needs to evaluate the integrals $I_{\xi,\tau_1;\xi',\tau_1'} = c_{\xi,\tau} c_{\xi',\tau_1'} \int_0^1 \beta^2 \times J_\nu(x_{\xi,\tau_1}\beta/\beta_W) J_{\nu'}(x_{\xi',\tau_1'}\beta/\beta_W)$. The transition strengths, when the contribution of the single particle is negligible ($t' = 0$), are shown in Fig. 1. Here, the $B(E2; J \rightarrow J') = |\langle J \parallel T^{(2)} \parallel J' \rangle|^2/(2J+1)$ values are shown. Again, all $B(E2)$ values are given in terms of only an overall scale, t. Generalization to $t' \neq 0$ is straightforward.

The five-dimensional square well with a five-dimensional spin-orbit interaction provides an example of the dynamical Bose-Fermi symmetry $Spin(5)$. For each $\xi = 1, 2, \ldots$, one has a series of multiplets characterized by the $Spin(5)$ quantum numbers $[\tau_1, \frac{1}{2}]$, with $\tau_1 = \frac{1}{2}, \frac{3}{2}, \ldots$. It is of interest to compare the spectrum and electromagnetic rates with those of the five-dimensional well without the spin-orbit interaction given in [10]. For each $\xi = 1, 2, \ldots$, one has a series of multiplets characterized by the $Spin(5)$ quantum numbers $[\tau, 0]$, with $\tau = 0, 1, \ldots$. The two solutions can be combined into a single solution by introducing the representations of the supergroup $OSp(5|4)$, which is then the dynamic supersymmetry of the problem. For each ξ, states are labeled by representations of $OSp(5|4)$. The five-dimensional square well was denoted in [10] by $E(5)$. The combined solution will be denoted here $E(5|4)$. Its spectrum is shown in Fig. 2. Supersymmetry relates the energy and $B(E2)$ scales for the even [panel (a)] and odd [panel (b)] systems. In particular, the (odd/even) ratio of energy scales is 1.066 and of $B(E2)$ scales is 1.121. Since the explicit construction of $OSp(5|4)$ cannot be done easily in the mixed realization coordinates-matrices employed here, but requires the introduction of Grassmann variables, its presentation will be postponed to a longer publication.

The solution of the five-dimensional square well coupled to a $j = 3/2$ particle can be used to study spectra of odd-even or even-odd nuclei at the critical value of the spherical to γ-unstable transition. However, even more here than in the case of even-even nuclei, the degeneracy of the τ mulitplets is broken by additional interactions and one thus needs, for a direct comparison with the data, a consideration of these interactions. In analogy with its algebraic counterpart, it is also possible here to introduce additional interactions which are diagonal in the chain $Spin(5) \supset Spin(3)$, through a three-dimensional spin-orbit interaction, $k' g(\beta)[2s \cdot \ell + \ell^2 + s^2]$. This interaction is diagonal with eigenvalue $\propto J(J+1)$. The solution remains the same but with $\Lambda = \tau(\tau+3) + k[\tau_1(\tau_1+3) + \frac{3}{4} - \tau(\tau+3)] + k' J(J+1)$. For $k = 1$, $\Lambda = \tau_1(\tau_1+3) + \frac{3}{4} + k' J(J+1)$. The solutions now depend on the parameter k'.

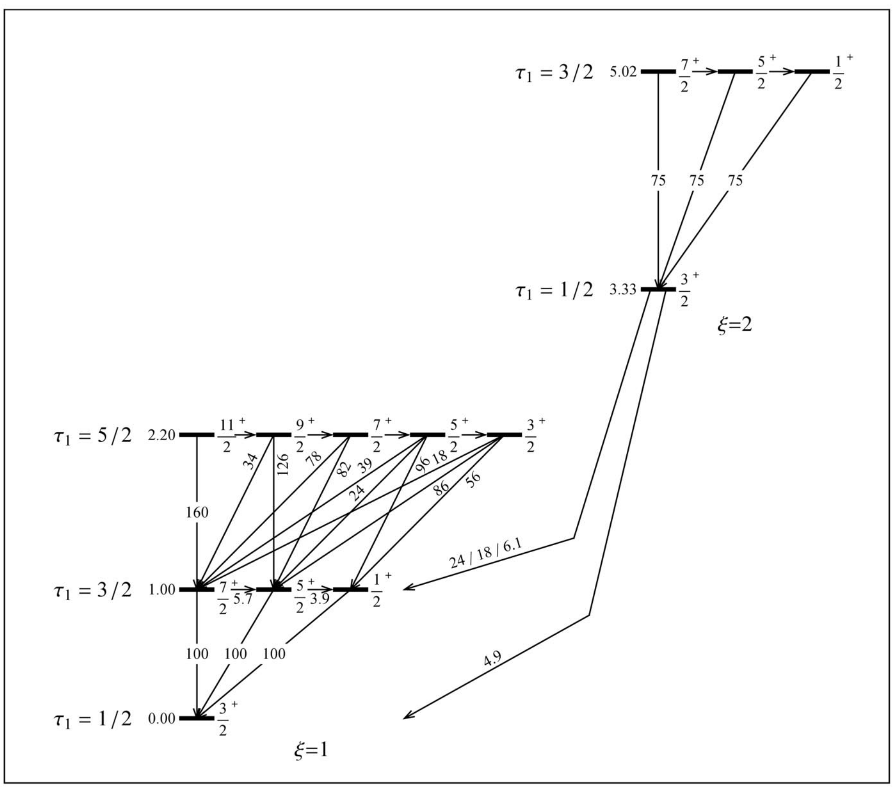

FIG. 1. The lowest portion of the spectrum of a $j = 3/2$ particle in a five-dimensional infinite square well with five-dimensional spin-orbit interactions. Energies are in units of the energy of the first multiplet, $\xi = 1$, $\tau_1 = \frac{3}{2}$. $B(E2)$ values are in units of the $B(E2)$ for the transition from the multiplet $\xi = 1$, $\tau_1 = \frac{3}{2}$ to the ground state $\xi = 1$, $\tau_1 = \frac{1}{2}$. A $d_{3/2}$ particle is shown. For $p_{3/2}$, all parities should be reversed.

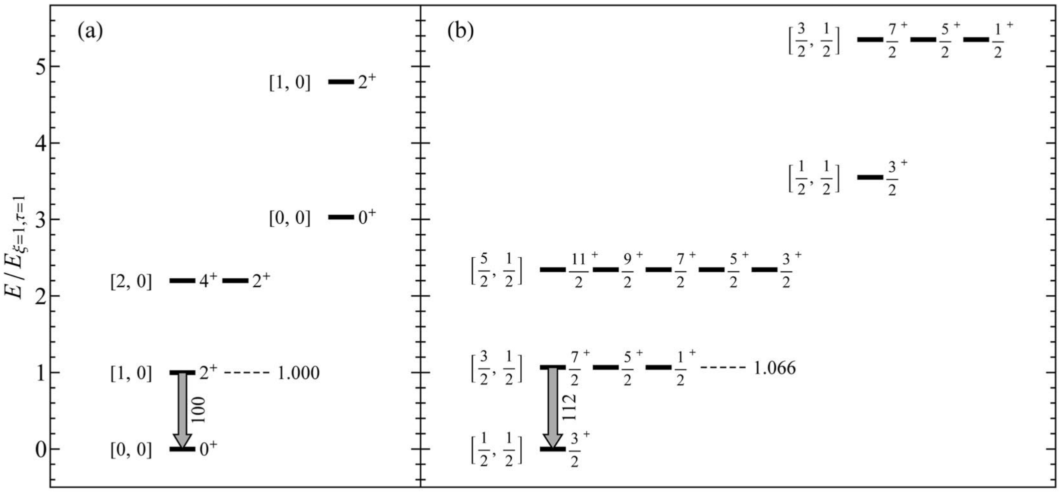

FIG. 2. The combined bosonic and fermionic spectra in $E(5|4)$. The energy scales for even-mass and odd-mass nuclei are related by supersymmetry, as are the $B(E2)$ scales (see text).

A search for experimental examples of $E(5|4)$ should concentrate on regions where the adjacent even-even nucleus is undergoing a spherical to γ-unstable transition, $E(5)$, and the single particle (proton or neutron) occupies a level with $j = 3/2$. An example of $E(5)$ has been found in ^{134}Ba [11]. States in ^{133}Ba built on the single particle neutron level $d_{3/2}$ are possible candidates. Also, in 1984, Bijker and Kota [12] showed that ^{63}Cu (an odd proton nucleus with the single particle in the $p_{3/2}$ level) is an example of $Spin(5)$ symmetry in which a $j = 3/2$ particle is coupled to a spherical even-even nucleus. The situation in this nucleus should be readdressed in light of the present development. The crucial measurement that distinguishes the two situations is that of the $B(E2)$ value connecting the state $\xi = 2$, $\tau_1 = 1/2$ with the ground state $\xi = 1$, $\tau_1 = 1/2$. In $E(5|4)$, this $B(E2)$ value is small (see Fig. 1), while in the case described in [12] its value is 100.

Finally, the method presented here can be used for other physical systems at a critical point of a second order shape phase transition. The solution for a three-dimensional square well with a spin $1/2$ particle can be used to study electrons in van der Waals molecules [13]. The Bose-Fermi symmetry of the three-dimensional equation is $Spin(3)$ and the supersymmetry is $OSp(3|2)$. The square well can be denoted by $E(3|2)$. Also, a situation similar in spirit to that discussed here, described by the group $OSp(1|2)$, was considered years ago by Balantekin *et al.* [14]. In all these applications, the compact supergroup $OSp(n|m)$ describes the degenerate multiplets (Fig. 2). Another use of $OSp(n|m)$ in nuclear spectroscopy was done years ago, but using the noncompact supergroup $OSp(n|m; R)$ to describe seniority supermultiplets [15].

The important new result of the present Letter is to have extended the concept of critical symmetry to critical supersymmetry, and, in doing so, provide a benchmark for the study of odd-even nuclei (and other mixed fermionic-bosonic systems) in the most difficult situation in which the system is undergoing a phase transition between two different phases (shapes).

This work was performed in part under DOE Grant No. DE-FG-02-91ER40608. I wish to thank Mark Caprio and Rick Casten for a critical reading of the manuscript.

[1] F. Iachello, Phys. Rev. Lett. **44**, 772 (1980).
[2] A. B. Balantekin, I. Bars, and F. Iachello, Phys. Rev. Lett. **47**, 19 (1981); Nucl. Phys. **A370**, 284 (1981).
[3] A. Metz *et al.*, Phys. Rev. Lett. **83**, 1542 (1999); Phys. Rev. C **61**, 064313 (2000); J. Gröger *et al.*, Phys. Rev. C **62**, 064304 (2000).
[4] F. Iachello and P. Van Isacker, *The Interacting Boson-Fermion Model* (Cambridge University Press, Cambridge, England, 1991).
[5] I. Bars, Physica (Amsterdam) **15D**, 42 (1985).
[6] B. F. Bayman and L. Silverberg, Nucl. Phys. **16**, 625 (1960).
[7] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, No. 14 (1952).
[8] J.-Q. Chen, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989), p. 228.
[9] F. Iachello and S. Kuyucak, Ann. Phys. (N.Y.) **136**, 19 (1981).
[10] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
[11] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85**, 3584 (2000).
[12] R. Bijker and V. K. B. Kota, Ann. Phys. (N.Y.) **156**, 110 (1984).
[13] F. Iachello and R. D. Levine, *Algebraic Theory of Molecules* (Oxford University Press, Oxford, England, 1991).
[14] A. B. Balantekin, O. Castaños, and M. Moshinsky, Phys. Lett. B **284**, 1 (1992).
[15] H. A. Schmitt, P. Halse, B. R. Barrett, and A. B. Balantekin, Phys. Lett. B **210**, 1 (1988).