

Unstable Fingering Patterns of Hele-Shaw Flows as a Dispersionless Limit of the Kortweg–de Vries Hierarchy

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We show that unstable fingering patterns of two-dimensional flows of viscous fluids with open boundary are described by a dispersionless limit of the Korteweg–de Vries hierarchy. In this framework, the fingering instability is linked to a known instability leading to regularized shock solutions for nonlinear waves, in dispersive media. The integrable structure of the flow suggests a dispersive regularization of the finite-time singularities.

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Introduction.—The Hele-Shaw cell is a narrow space between two parallel planes—a device used to study the 2D dynamics of a fluid with an open boundary. Air (regarded as an inviscid fluid) occupies a bounded domain of the cell, otherwise filled by a viscous incompressible fluid. When more air is injected through a well, the free boundary evolves in a complicated and unstable manner. In finite time, an arbitrary smooth initial boundary develops a pattern of branched fingers. This mechanism is generic for many growth problems [1].

Starting from early works [2], this fingering instability was linked to the theory of analytical functions and 2D conformal maps. Recently, a connection with fractal self-similar patterns of stochastic growth, like diffusion-limited aggregation (DLA), has been recognized [3]. The Hele-Shaw problem has also emerged in the context of electronic physics in low dimensions. For 1D electrons, the Fermi sphere is deformed by external perturbations according to the Hele-Shaw mechanism. The same is true for deformations of an electronic droplet in a quantizing magnetic field [4]. Similarly, the Hele-Shaw problem is connected to statistical ensembles of normal or complex random matrices, where a non-Gaussian potential deforms the support of eigenvalues [5].

In this Letter we emphasize yet another connection, which may explain the occurrence of this phenomenon in different physical situations. We show that the fingering instability is linked to the integrable Korteweg–de Vries (KdV) hierarchy of differential equations. More precisely, we show that developed fingers, in the absence of surface tension, are described by the dispersionless KdV hierarchy. Hele-Shaw fingers relate to the same solution of the KdV hierarchy as that exploited in 2D gravity and Hermitian random matrix ensembles [6]. The physics of interface dynamics gives a geometrical interpretation to algebraic objects used in dispersionless soliton equations [7].

The integrable structure of the Hele-Shaw problem has been observed in our earlier papers [8]. There, we have

shown that the problem is equivalent to the dispersionless limit of the Toda integrable hierarchy. This is an exact relation. Here we concentrate on the critical (turbulent) regime of the flow, where fingers are close to a cusplike singularity. In this case, one may concentrate only on the vicinity of the tip of a single finger, neglecting the rest of the boundary. The KdV integrable structure emerges in this regime. It can be obtained as a reduction of the Toda hierarchy; here we derive it directly from hydrodynamics of the critical regime.

Darcy's law.—Consider a 2D domain (a “bubble”) occupied by “air” regarded as an incompressible fluid with low viscosity. The rest of the cell is occupied by another incompressible fluid but with high viscosity. Air is injected into the cell through the origin, at a constant rate, while the viscous fluid is evacuated at “infinity” (edges of the cell). The area of the bubble is proportional to time t , normalized to πt . The Navier-Stokes equation in the 2D geometry gives a simple rule for the dynamics of the boundary γ : velocity of the boundary is proportional to the gradient of pressure,

$$\vec{v} = -\vec{\nabla}p. \quad (1)$$

Pressure is harmonic for incompressible fluids. Inside the bubble, it is constant (set to $p = 0$), due to low viscosity. In the absence of surface tension, it is continuous across the boundary, and hence a solution of the Dirichlet problem

$$\Delta p = 0, \quad p|_{\infty} \rightarrow -\log|z|, \quad p|_{\gamma} = 0. \quad (2)$$

The origin of the fingering instability is intuitively clear—high curvature portions of the boundary move with a higher velocity than the rest and get even more curved.

Problem (2) is solved by potential theory. Consider a holomorphic function $\phi(z) = \xi(x, y) + ip(x, y)$, whose imaginary part is pressure. The function $\phi(z)$ is a univalent conformal map of the exterior of the bubble to the cylinder $\text{Im}\phi > 0$, and its derivative $\partial\phi(z)$ taken on the boundary is the conformal measure of the boundary. Denote the inver-

sion map $z(\phi)$. Darcy's law (1) reads as follows: the complex velocity of the boundary is proportional to the conformal measure.

For simplicity, let the bubble and finger have a symmetry axis. In Cartesian coordinates, the boundary can be described by a multivalued function $y(x)$, Fig. 1.

Another useful way to describe the boundary is through the height function $h(z)$. It is an analytic function in some vicinity of the boundary, taking values $h(x + iy) = y(x)$ on the boundary. From the Schwarz reflection principle, $h(\phi) - \overline{h(-\bar{\phi})} = -i[z(\phi) - \overline{z(-\bar{\phi})}]$. Therefore,

$$2ih(\phi) = z(\phi) - \overline{z(-\bar{\phi})}. \quad (3)$$

We also use a generating function $\partial_z \omega(z) = -2ih(z)$. The complex velocity $v = \dot{z}$ can be analytically continued into the outer domain, as $v = \partial_t \partial_z \omega(z) = -2i\partial_t h(z)$. Then from $v = -2i\partial_z \phi$, Darcy's law becomes

$$\partial_t h(z) = \partial_z \phi(z). \quad (4)$$

Branch points.—If the boundary is analytic, the map $\phi(z)$ and the height function $h(z)$ can be analytically continued inside the bubble, until they reach singularities. For an important class of analytic boundaries, the conformal map $z(\phi)$ is a rational function of $e^{i\phi}$. Such domains are called algebraic. In this case, the singularities of the inverse map are generically simple branch points $z_0(t), \dots, z_{q-1}(t)$ (analogs of Riemann invariants in hydrodynamics), where $\partial_z \phi \sim [z - z_i(t)]^{-1/2}$. Hele-Shaw flows preserve the class of algebraic domains [9].

As the boundary moves, so do the branch points. We identify the traces of the branch points with the branch cuts. Comparing singularities of (4), we get a system of coupled equations for the branch points,

$$\dot{z}_i = \frac{\partial \phi}{\partial h} \Big|_{z_i}, \quad i = 0, 1, \dots, q-1. \quad (5)$$

Finite-time singularities.—In full generality, these equations are difficult to analyze. Qualitatively, branch points tend to move towards the boundary, which is pushed away, forming a finger. When, eventually, a branch point reaches the boundary, the finger forms a cusplike singularity. For almost all initial algebraic domains, a cusplike singularity emerges at a finite time, i.e., finite area of the bubble. The

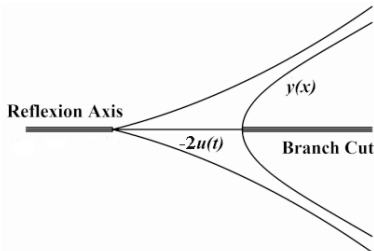


FIG. 1. A finger extends to the right of its boundary $y(x)$. The distance from the tip to the singular point is $-2u(t)$.

idealized law (1) then needs to be corrected by regularization [10]. Some examples of singular Hele-Shaw flows were elaborated in [11].

In this Letter we suggest a unified approach to the finite-time singularities based on a singular limit of the KdV integrable hierarchy, appropriate when fingers are close to a cusp. Generically, an isolated branch point, $z_0 = (x_0, 0)$, is found very close to the boundary (tip of the finger), i.e., $\phi(z_0) \approx 0$. From (5), this branch point (and tip) moves with velocity $\dot{x}_0 = \partial \phi / \partial y|_{x_0}$.

The origin of the finite-time singularities can already be seen from this equation by means of scaling analysis. Around the tip, the map $z(\phi)$ has a regular expansion $z(\phi) - x_0 = ia\phi + b\phi^2 + ic\phi^3 + \dots$, with real a, b, c, \dots . It follows from the reflection formula (3) that $h(\phi) = a\phi + c\phi^3 + \dots$. If a branch point is close to the tip, then a is small, while b and c are of order 1. If there is only one scale, $u(t)$, then x_0 , measured from what will be the cusp tip, and a are of the same order $u(t)$. The velocity of the finger tip $\dot{x}_0 \sim \dot{u}$ grows with the curvature $a^{-1} \sim u^{-1}$ of the tip such that $\dot{u} \sim u^{-1}$, and hence the scale vanishes as $u \sim (t_c - t)^{1/2}$. At the critical time $t = t_c$, $u = 0$ and the curve forms a (2, 3) cusp, $y(x) \sim x^{3/2}$.

More general cusps are characterized by two integers (q, p) , $q < p$, implying that the finger is bounded by $|y(x)| \sim x^{p/q}$, $x > 0$, q is even. Higher cusps correspond to higher order singularities merging at the boundary. Here we consider the case $q = 2$. The details on the higher cusps will be given elsewhere.

Critical regime and scaling function.—When the finger is very close to becoming a cusp, the rest of the bubble does not affect its evolution. We call this regime *critical*. In the critical regime, different scales separate. It is convenient to define the scales in the complex plane of ϕ . At $\phi = O(u^{1/2})$ we see the details of the tip. At $1 \gg \phi \gg u^{1/2}$, the details of the tip cannot be seen, and the finger looks like a cusp. Finally, $\phi = O(1)$ corresponds to the rest of the body of the bubble. The scale $u(t)$ changes with time and eventually disappears at the critical point. In that limit, the finger is scale invariant.

To summarize, in the critical regime $u \rightarrow 0$, $t \rightarrow t_c$, the inverse map and the height function are

$$z(\phi) - ih(\phi) = u^{q/2} Q\left(\frac{\phi}{\sqrt{u}}\right), \quad h(\phi) = u^{p/2} P\left(\frac{\phi}{\sqrt{u}}\right), \quad (6)$$

where the scaling functions Q and P do not depend on u . At $q = 2$, Q is an even quadratic polynomial and the scaling ansatz holds everywhere in the critical region.

The asymptotic behavior of the finger, $y \sim \pm x^{p/q}$, allows us to identify the scaling functions by matching asymptotes in different regions: far from the tip, the finger asymptote is $y \sim \pm x^{p/q}$. Since $y \ll x \ll 1$, we approximate $y \sim z^{p/q}$, where $z = x + iy$. Higher order corrections

will make y real. Therefore, we conclude that the height function, in the domain close to the boundary, but away from the tip, is $h(z) \sim z^{p/q}$. This also means that at $u^{1/2} \ll \phi \ll 1$, and close to the real axis, $Q(\phi) \sim \phi^q$, $P(\phi) \sim \phi^p$. Close to the tip, where $\phi \ll u^{1/2}$, both $z(\phi)$ and $h(\phi)$ are regular in ϕ . The only holomorphic functions matching these conditions are polynomials of degrees q and p , respectively, with real coefficients.

Remarks.—Approximating the map $z(\phi)$ by polynomials makes it no longer univalent at $q > 2$ [Eq. (6)]. It now covers the z plane q times. This means that at $q > 2$ the scaling ansatz does not hold everywhere in the upper half-plane of ϕ . It holds only in certain sectors of the plane adjacent to the real axis and breaks down otherwise. Their union, covering roughly the $2/q$ part of the upper half-plane—a *physical domain*—determines a physical branch of the map $\phi(z)$. The physical branch must obey the reflection symmetry and be real at the boundary of the finger. This is impossible unless the stream function $\text{Re}\phi$ is allowed to have a discontinuity across the reflection axis outside the finger. Every monomial $z^{n/q}$ in the asymptotic expansion $\phi(z) = z^{1/q} + \text{negative powers of } z^{1/q}$ should be understood as $z^{1/q}$, if $0 < \arg z < \pi - \epsilon$, and $e^{2\pi i/q} z^{1/q}$, if $-\pi + \epsilon < \arg z < 0$. As a result, the imaginary part of $\phi(z)$ has a finite discontinuity on the reflection line, at least far away from the tip, unless $q = 2$.

This indicates that cusps with $q > 2$ do not appear for algebraic simply connected domains. We do not have an interpretation of this phenomenon. The nonunivaliddness of the map suggests that there is another finger at the left, while the cusp singularity implies their merging.

dKdV hierarchy.—From (6), in the scaling limit, the generating function has the asymptotic expansion $\omega(z) - \omega(z_0) = -2i \int_{z_0}^z h(z') dz' + O(u^p) \approx u^{(p+q)/2} \int_0^\phi P dQ$. It is a truncated Laurent series in $z^{1/q}$,

$$\omega(z) = i \sum_{n=1}^{p+q} t_n z^{n/q} + \text{negative powers of } z^{1/q}. \quad (7)$$

We will see in a moment that all the coefficients t_k except t_1 are conserved. Moreover, the coefficient t_1 is proportional to time, measured from the moment of singularity. The coefficients t_k and t_1 are called *deformation* and *evolution* parameters, respectively. The generating function and the height function can be expressed through deformation parameters. As polynomials of ϕ , they are

$$\omega = i \sum_{n=1}^{p+q} t_n \omega_n, \quad h = \sum_{n=1+q}^{p+q} \frac{n}{q} t_n \omega_{n-q}. \quad (8)$$

Here $\omega_n(\phi)$ is the polynomial part of $u^{n/2} Q^{n/q}(\phi)$.

Now we can describe the evolution of the curve with respect to all the deformation parameters. The arguments are borrowed from [7] with little changes. We note that $\partial_{t_n} \omega$ is a polynomial of degree n in ϕ . As a Laurent series

in $z^{1/q}$, it has only one term, $z^{n/q}$, with a positive degree. The only polynomial of this kind is ω_n . Therefore,

$$\partial_{t_n} h(z) = \partial_z \omega_n(z). \quad (9)$$

Notice that the flow in real time (4) appears on the same footing as flows with respect to the deformation parameters. Setting $n = 1$, we recover the flow Eq. (4).

Equations (4) and (9) can be cast in the form of flow equations, if one takes the time derivative at constant ϕ . Defining the Poisson brackets with respect to the canonical pair t, ϕ such that $\{f, g\} = \partial_t f \partial_\phi g - \partial_t g \partial_\phi f$, Eqs. (4) and (9) read $\{h, z\} = 1$ and $\partial_{t_n} h(\phi) = \{h, \omega_n\}$. From this we conclude that the inverse map evolves as

$$\partial_{t_n} z = \{z, \omega_n\}. \quad (10)$$

Compatibility of these equations gives a closed set of nonlinear equations for the coefficients of the polynomials (6) $\partial_{t_m} \omega_n - \partial_{t_n} \omega_m = \{\omega_n, \omega_m\}$. At $q = 2$, it is the dispersionless KdV hierarchy (dKdV). At $q > 2$, Eq. (10) constitutes the dispersionless Gelfand-Dikii hierarchy [7].

Solutions of the dKdV hierarchy.—At $q = 2$, a complete solution is available. In this case $z(\phi) = \phi^2 - 2u$, the evolution does not depend on t_{2n} and

$$\omega_{2n+1} = \sum_{k=0}^n \frac{(2n+1)!!}{(2n-2k+1)!!} \frac{(-u)^k}{k!} \phi^{2n-2k+1}. \quad (11)$$

Equation (10) becomes

$$\partial_{t_{2n+1}} u = \frac{(2n+1)!!}{n!} (-u)^n \partial_{t_1} u.$$

The first equation is the familiar Hopf-Burgers equation

$$\partial_{t_3} u + 3u \partial_{t_1} u = 0. \quad (12)$$

The hodograph transformation gives a general solution

$$\sum_{k=0}^l \frac{(2k+1)!!}{k!} t_{2k+1} (-u)^k = 0, \quad p = 2l + 1. \quad (13)$$

More details of the application to the singular limit of the Hele-Shaw flow can be found in [12].

KdV hierarchy.—Equation (12) is a singular $\hbar \rightarrow 0$ limit of the full dispersive KdV equation

$$4\partial_{t_3} u + 12u \partial_{t_1} u + \hbar^{5/2} u_{ttt} = 0. \quad (14)$$

To clarify the nature of this limit, we recall the definition of the q -reduced Kadomtsev-Petviashvili hierarchy. It is a set of nonlinear equations compactly written through a pair of operators L, M , differential operators in time of degrees q and p :

$$L = \hat{\phi}^q - \sum_{l=1}^q [e_l, \hat{\phi}^{q-l}]_+, \quad M = \sum_{n=1+q}^{p+q} \frac{n}{q} t_n \Omega_{n-q}. \quad (15)$$

Here $[\]_+$ denotes the anticommutator, $\hat{\phi} = \hbar \partial_{t_1}$, and the coefficients e_l are functions of a string of “times”

t_1, \dots, t_{p+q} , and $\Omega_n = L_+^{n/q}$ is a differential operator of degree n , obtained from the pseudodifferential operator $L^{n/q}$ by omitting a nondifferential part. The dependence of the coefficients e_l on the times is introduced by the Lax-Sato equations $\hbar \partial_{t_k} L = [L, \Omega_n]$, $\hbar \partial_{t_k} M = [M, \Omega_n]$, supplemented by the condition $[L, M] = \hbar$. An alternative way to define the hierarchy is to impose the consistency conditions $\partial_{t_n} \Omega_m - \partial_{t_m} \Omega_n = [\Omega_m, \Omega_n]$.

The dispersionless limit is obtained by replacing the differential operator $\hat{\phi} = \hbar \partial_t$ by a function $\hat{\phi} \rightarrow \phi$, and the commutator $i\hbar^{-1}[\cdot, \cdot]$ by the Poisson bracket $\{\cdot, \cdot\}$. The objects of the Lax-Sato theory are expressed in terms of the Hele-Shaw flow: $\hat{\phi}$, L , and M become, respectively, the complex potential ϕ , the coordinate z , and the height function h , while \hbar is a quantum of area.

Applications.—The integrable structure revealed in this Letter allows using the KdV theory in the study of moving boundary problems. Special solutions of the hierarchy describe bubble coalescence, bubble breakoff, branching, etc. We report some of them in [12].

It is known that the dispersionless limit of nonlinear waves is singular [13], as smooth initial data evolve into a multivalued (“overhanged”) function within a finite time. These solutions are equivalent to the finite-time singularities of the Hele-Shaw flow. The integrable structure of singularities suggests a novel, “dispersive” regularization. The flow (1) is an ill limit of the true dispersive flow, just as Eq. (12) is a singular limit of Eq. (14). We will present details of this regularization elsewhere. Here we note only that this regularization treats the Hele-Shaw flow as a stochastic process of deposition of small particles with an area \hbar , as the DLA [3].

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