Vorticity Cutoff in Nonlinear Photonic Crystals

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(Received 8 November 2004; published 18 July 2005)

Using group-theory arguments, we demonstrate that, unlike in homogeneous media, no symmetric vortices of arbitrary order can be generated in two-dimensional (2D) nonlinear systems possessing a discrete-point symmetry. The only condition needed is that the nonlinearity term exclusively depends on the modulus of the field. In the particular case of 2D periodic systems, such as nonlinear photonic crystals or Bose-Einstein condensates in periodic potentials, it is shown that the realization of discrete symmetry forbids the existence of symmetric vortex solutions with vorticity higher than two.

DOI: 10.1103/PhysRevLett.95.043901 PACS numbers: 42.70.Qs, 03.75.Lm, 42.65.Tg

Vortices are complex waves characterized by a phase singularity, a point where phase is not defined and consequently vortex amplitude vanishes [1]. This feature is shared by all complex waves and appears in many different nonlinear systems, ranging from fluid dynamics to photonics. The most prominent characteristic of a vortex is that energy flows around the phase singularity in such a way that its flux is quantized. This flux quantization is reflected in a dislocation of the vortex phase which changes as an integer multiple of 2π when one performs a complete tour around the singularity. This integer number is referred to as vorticity (also known as winding number, "topological charge", or spin). An optical vortex with a rotationally invariant amplitude in a nonlinear Kerr medium, experimentally observed in homogeneous self-defocusing media [2], can be understood as an eigenmode of the equivalent rotationally invariant waveguide generated by itself [3]. Thus, a vortex appear as an object carrying well-defined angular momentum: $\phi_l = e^{il\theta}f(r)$. In this case, angular momentum and vorticity are the same integer number; a consequence of the continuous $O(2)$ symmetry of the operator defining the equivalent waveguide. However, in systems such as 2D nonlinear photonic crystals or Bose-Einstein condensates in 2D periodic traps this $O(2)$ symmetry is replaced by a discrete-point symmetry. Angular momentum is no longer well defined and thus the angular momentum-vorticity equivalence is lost. Nevertheless, optical vortices have been predicted to exist in 2D periodic photonic crystals [4,5] and in photonic crystal fibers [6], and experimentally observed in optically-induced photonic lattices [7]. Although these solutions cannot any longer have well-defined angular momentum, certainly all of them present neat phase dislocations that can be characterized by an integer vorticity value. In this Letter, we will prove how to reinterpret vorticity in terms of the rotational properties of vortex solutions without resorting to the angular-momentum concept. As a result, severe restrictions on vorticity values will be found using group-theory arguments.

Let us consider the following general nonlinear equation for stationary states:

$$[L_0 + L_{\rm NL}(|\phi|)]\phi(x, y) = -\mathcal{E}\phi(x, y), \tag{1}$$

where L_0 is a linear field-independent self-adjoint operator (normally dependent on gradients and functions of the transverse coordinates) and $L_{\rm NL}(|\phi|)$ is the nonlinear field-dependent piece of the full operator acting on the field ϕ. This equation is valid for all types of 2D systems in which the nonlinearity depends on the field through its modulus. Many different systems can be modeled using an equation that can be written in the form given by Eq. (1). We are interested in systems that, besides being described by Eq. (1), are invariant under some discrete-symmetry group G: $[L, G] = 0$ ($L \equiv L_0 + L_{\rm NL}$). This means that we assume that all linear and nonlinear coefficients appearing in the operators defining Eq. (1) are invariant under the G group. Our goal is to study the implications that the realization of discrete symmetry has on the characterization of vortex solutions of Eq. (1).

The key concept in our approach is the so-called group self-consistency condition. This condition establishes that if a system described by Eq. (1) is invariant under some discrete-symmetry group G then any of its solutions either belongs to one representation of the group G or to one of its subgroups G' ($G' \subset G$) [8]. Note that the identity group is

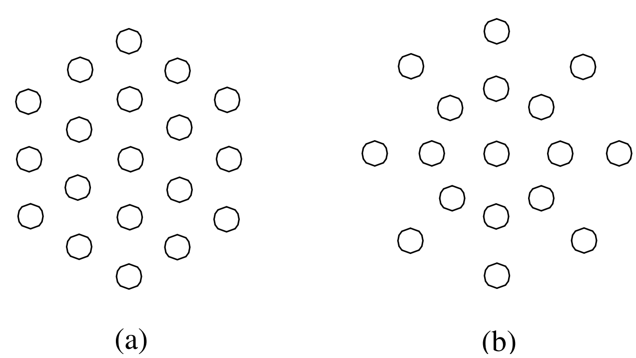

FIG. 1. Two examples of structures invariant under $2\pi/n$ rotations plus specular reflections on the x and y axis: (a) sixfold rotation axis (C_{6v} group), and (b) eightfold rotation axis (C_{8v} group).

always a subgroup of any group and, therefore, asymmetric solutions also satisfy the group self-consistency condition [5]. In this Letter, however, we will focus on symmetric solutions exclusively.

The elements of point-symmetry groups in a plane are rotations through integral multiples of $2\pi/n$ about some axis (called an n-fold rotation axis), reflections on a mirror plane containing the axis of rotation and combinations of both. Groups containing an n-fold rotation axis constitute the C_n groups. When, in combination with the n-fold rotation axis, these groups have mirror planes, one generates the so-called C_{nv} groups. In Fig. 1 we give two examples of structures exhibiting C_{6v} and C_{8v} point symmetries.

We will prove next how vorticity is affected by the finite order of the n-fold rotation axis defining the C_n (or C_{nv}) group. In order to do so, we need first to properly characterize the different representations of a C_n group. Since C_n groups are Abelian, its representations are one-dimensional and given by a single scalar complex number (the character of the representation) [9]. This scalar is nothing but a root of unity of order n and thus the representations of the C_n group are given by $\{1, \epsilon^{\pm 1}, \ldots, \epsilon^{\pm l}, \ldots, \epsilon^{n/2}\}$ for even n and $\{1, \epsilon^{\pm 1}, \ldots, \epsilon^{\pm l}, \ldots, \epsilon^{\pm(n-1)/2}\}$ for odd n, where $\epsilon = \exp(2\pi i/n)$. In Fig. 2 we present, as an example, the construction of the roots of unity for the C_6 (even n) and C_3 (odd n) groups. Each representation can be labeled by the natural number l and, when present, by its sign. We denote it by $\mathcal{D}_{l,s}$ ($l \in \mathbb{N}$, $s = \pm$). No sign is needed for the identity representation $\mathcal{D}_0$ ($l = 0$) nor for $\mathcal{D}_{n/2}$ ($l = n/2$, even n). A state belonging to representations with $l \neq 0$, $n/2$ can be written as $|l, s\rangle$ with $0 < l < n/2$ (if n is even) or $0 < l \leq (n-1)/2$ (if n is odd). When we act with a group operator G (representing a discrete rotation of angle $2\pi/n$) on a function belonging to a representation $\mathcal{D}_{l,s}$, it transforms as $G\phi_{\bar{l}} = \epsilon^{\bar{l}}\phi_{\bar{l}}$, where $\bar{l} \equiv sl$ ($s = \pm$, $l \in \mathbb{N}$). Clearly too, $G\phi_0 = \phi_0$ and $G\phi_{n/2} = \epsilon^{n/2}\phi_{n/2}$ (even n). If there is no other symmetry involved, $[L, G] = 0$ implies that every one-dimensional representation is characterized by a different L eigenvalue: $L\phi_{\bar{l}} = -\mathcal{E}_{\bar{l}}\phi_{\bar{l}}$. Representations are thus nondegenerated.

We proceed now to explicitly construct functions belonging to $l \neq 0$ representations. Let us consider the complex coordinate vector $u = x + iy = re^{i\theta}$. Integer powers of u have well-defined transformation properties under a $2\pi/n$ rotation: $u^{\bar{l}\theta \to \theta + 2\pi/n} \to \epsilon^{\bar{l}}u^{\bar{l}}$. Therefore, we can easily construct a function in the $\mathcal{D}_{l,s}$ representation of C_n as

$$\phi_{\bar{l}}(u) = u^{\bar{l}}\phi_0^{(\bar{l})}(u), \tag{2}$$

$\phi_0^{(\bar{l})}$ being a function in the $\mathcal{D}_0$ representation of C_n. Clearly, $G\phi_{\bar{l}} = \epsilon^{\bar{l}}\phi_{\bar{l}}$.

The representations of C_{nv} (discrete rotations plus reflections) are easily obtained from those of C_n groups [9]. The existence of the extra symmetries provided by mirror reflections yields to degeneracies for high-order ($\bar{l} \neq 0$) representations. These states are now doubly degenerated; they form pairs of complex-conjugated functions (ϕ_l, ϕ_l^*) with the same L eigenvalue: $L\phi_{\bar{l}} = -\mathcal{E}_l\phi_{\bar{l}}$. Remarkable exceptions are the $\mathcal{D}_0$ and $\mathcal{D}_{n/2}$ representations. Because of their different behavior under mirror reflections there are two distinct nondegenerated one-dimensional $l = 0$ representations: $|0; ++\rangle$ and $|0; --\rangle$. They transform differently under reflections with respect to the x and y axis: $R_{x,y}|0; ++\rangle = +|0; ++\rangle$ and $R_{x,y}|0; --\rangle = -|0; --\rangle$. The $|0; ++\rangle$ state has maximal symmetry. Fundamental solitons belong to this identity representation of C_{nv}. In the same way, there are also two different nondegenerated one-

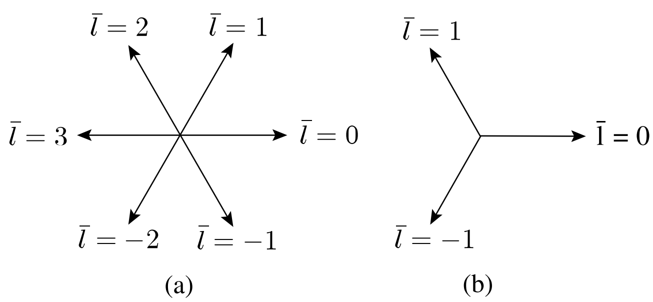

FIG. 2. Roots of unity diagrams displaying the representations of: (a) C_6 and (b) C_3.

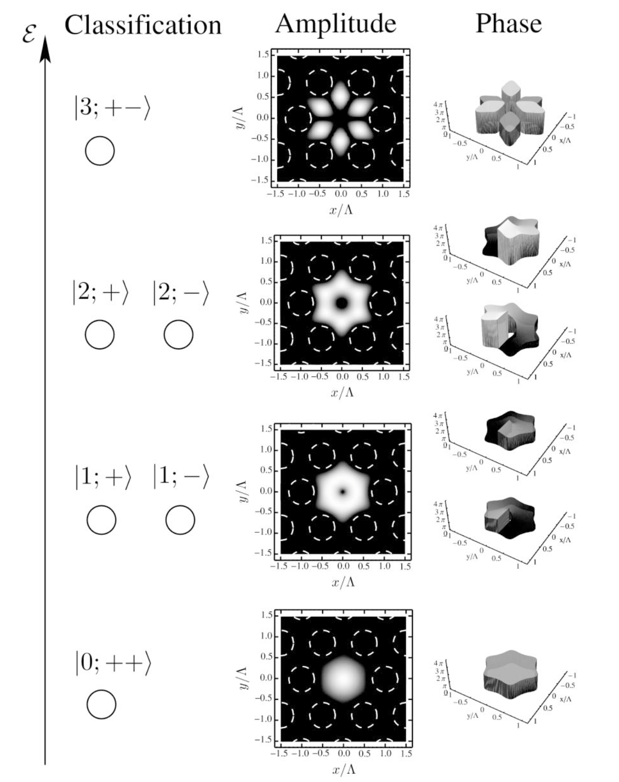

FIG. 3. Lowest order eigenfunctions of a nonlinear operator L generated by a soliton solution in the identity (fundamental) representation of C_{6v}. The symmetry of the full operator is C_{6v}: $[L, C_{6v}] = 0$.

dimensional representations with $l = n/2$ (even n): $|n/2; +-\rangle$ and $|n/2; -+\rangle$. The distinction is made by $R_{x,y}$ reflections. In Fig. 3 we show the lowest order eigenfunctions of the spectrum of a C_{6v}-invariant operator self-consistently generated by a fundamental soliton solution ($\phi_{\mathrm{fund}} = \phi_{0,++}$): $L = L_0 + L_{\mathrm{NL}}(|\phi_{\mathrm{fund}}|)$ (see the final section of the Letter for details on the physical system associated to L). We easily recognize, from lower to higher values of $\mathcal{E}$, the $|0; ++\rangle$ self-consistent state (i.e., the fundamental soliton), the doubly degenerated $|1; \pm\rangle$ and $|2; \pm\rangle$ states and the nondegenerated $|3; +-\rangle$ state. The rest of the spectrum, including continuum delocalized states, systematically falls into the representations described above.

Vorticity v can be defined as the integer variation (in 2π units) that the phase of a complex field experiments under a 2π rotation around a rotation axis. Solutions with nonzero vorticity are called vortices of order v. They are characterized by their rotation axis, whose intersection with the 2D plane defines the vortex center, where their amplitude vanishes. If $\Phi(r, \theta)$ represents the phase of a complex vortex field of order v given by $f_v = |f_v|e^{i\Phi}$, then $\Phi(r, \theta + 2\pi) - \Phi(r, \theta) = 2\pi v$, where the polar coordinates are referred to a reference frame centered on the rotation axis. For systems enjoying a 2D point symmetry, this axis is naturally given by the n-fold rotation axis of the corresponding C_n (or C_{nv}) group.

According to the group self-consistency condition, all symmetric solutions of Eq. (1) in a system with C_n symmetry have to lie on the representations of C_n or of any of its subgroups. Let us consider now a solution $\phi_{\bar{l}}$ in the $\mathcal{D}_{l,s}$ representation of C_n given by Eq. (2). Its phase will be given by $\arg\phi_{\bar{l}}(r, \theta) = \bar{l}\theta + \arg\phi_0^{(\bar{l})}(r, \theta)$. Since $\phi_0^{(\bar{l})}(r, \theta)$ is invariant under rotations, $\arg\phi_{\bar{l}}(r, \theta + 2\pi) = \arg\phi_{\bar{l}}(r, \theta) + 2\pi\bar{l}$. Therefore we find the important relation between the index representation and vorticity:

$$v = \bar{l}. \tag{3}$$

Vortices are thus solutions belonging to $\mathcal{D}_{l,s}$ representations with $l \neq 0$. There is, however, no vortex associated to $l = n/2$ (even n). It can be proved that $\phi_{n/2}$ is a real field, so that its argument is a function that can only take the values 0 or π. More explicitly, from Eq. (2), $\phi_{n/2} \sim \cos[n\theta/2 + \arg\phi_0^{(n/2)}(r, \theta)]$, which has the phase behavior of alternating signs typical of a nodal soliton and not of a vortex [8]. In C_{nv}, the behavior of the $\phi_{n/2,+-}$ and $\phi_{n/2,-+}$ functions is also of the nodal-soliton type, as one can check by observing the phase of the $|3; +-\rangle$ state in Fig. 3.

Let us summarize now our main conclusions. First, if a system is invariant under a C_n or C_{nv} point-symmetry group, the solutions of Eq. (1) belong to representations of these groups or of their corresponding subgroups. Second, symmetric solutions of Eq. (1) are characterized by the representation index l, which has an upper bound fixed by the order of the group: $l \leq n/2$ (even n) and $l \leq (n-1)/2$ (odd n). Third, the vorticity v of the vortex solutions of such a system has a cutoff due to Eq. (3) and the upper bound for l:

$$|v| < n/2(\text{even } n) \quad \text{and} \quad |v| \leq (n-1)/2(\text{odd } n). \tag{4}$$

Note that the group of continuous rotations on a plane can be understood as the limiting case $O(2) = \lim_{n\to\infty} C_n$ and, thus, Eq. (4) correctly establishes the absence of a cutoff for it ($|v| < \infty$).

When we deal with 2D periodic systems, the realization of discrete symmetry has particular features as it is well-known in crystallography [10]. The crystal structure is constructed according to a pattern that repeats itself to "tessellate" the 2D plane in such a way that only patterns that exhibit a selected set of symmetries can satisfy this property. The important result for us here is that pattern periodicity establishes a restriction on the order of discrete rotations allowed in plane groups. Only n-fold rotations of order 2, 3, 4, and 6 are permitted in a 2D periodic crystal [10].

The previous group analysis has important implications for 2D nonlinear periodic systems. The maximum n-fold rotation symmetry compatible with periodicity is a sixth-fold rotation, which means that the maximum value for the order n of C_n and C_{nv} point-symmetry groups in 2D periodic systems is $n = 6$. Consequently, the point-symmetry group of a solution cannot exceed this order: $n \leq 6$. Since vorticity is restricted by the order of the point-symmetry group according to Eq. (4), we come up to the conclusion that in 2D nonlinear periodic systems of the type described by Eq. (1) vorticity has a strict bound: $|v| \leq 2$. Putting this into words, there are no vortices of order higher than 2 in 2D nonlinear periodic systems described by Eq. (1).

In order to illustrate our previous theoretical results, we have numerically studied a realistic system, namely, a photonic crystal fiber (PCF). A PCF is a type of 2D photonic crystal consisting on a regular lattice of holes in silica (characterized by the hole radius a and the lattice period or pitch Λ) extending along the entire fiber length. When one considers that the silica response is nonlinear (nonlinearity represented by the nonlinear coefficient γ, defined in Ref. [6]), a PCF becomes a 2D nonlinear photonic crystal. The nonlinear propagation modes of a PCF for monochromatic illumination in the scalar approximation verify Eq. (1) with $\mathcal{E} = -\beta^2$, β being the mode propagation constant (see [6]). Among possible hole-distribution geometries we choose that based on a triangular lattice with C_{6v} symmetry [see Fig. 1(a)]. The reason of our symmetry choice is simple. As proved before, the C_{6v} group provides the highest vorticity solutions since it corresponds to the maximal point symmetry achievable in a 2D nonlinear photonic crystal. Note that although we simulate an ideally infinite structure, the group-theory results equally apply to finite size systems. In Fig. 4 we find the first three (from lowest to highest value of β^2)

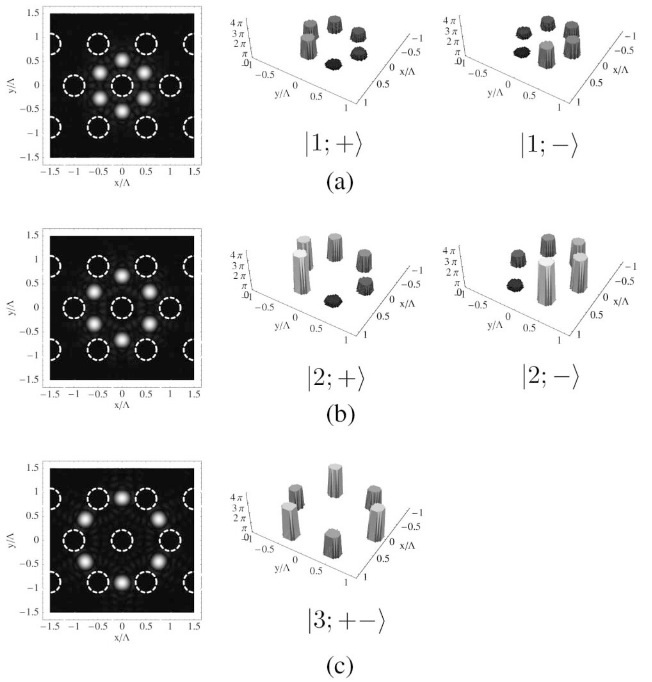

FIG. 4. Higher-order solitons for a periodic C_{6v} PCF: (a)–(b) First- and second-order vortex pairs, $|1;\pm\rangle$ and $|2;\pm\rangle$; (c) nodal soliton of order 3, $|3;+-\rangle$.

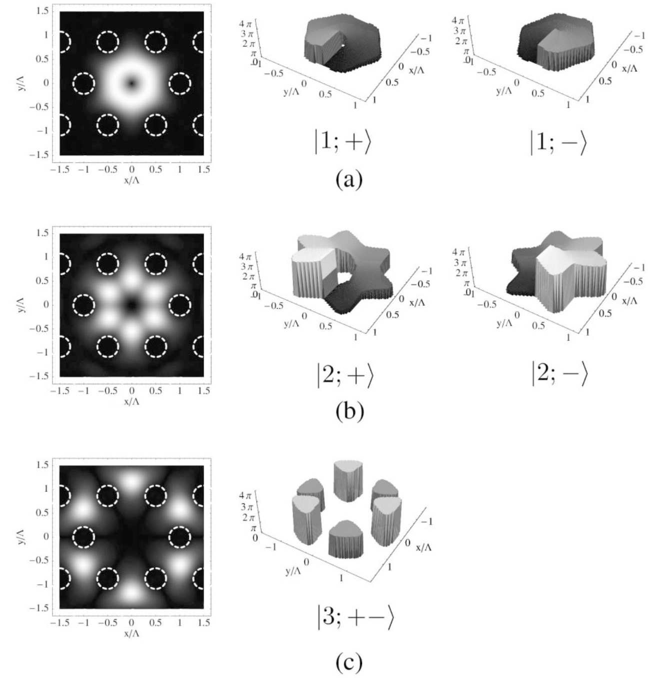

FIG. 5. Same as in Fig. 4 but in a PCF with defect.

higher-order solitons of a perfectly periodic PCF (without defect) calculated for the values $a = 5$ μm, $\Lambda = 26$ μm, and $\lambda = 1064$ nm at $\gamma = 0.01$. In Fig. 5 we present the same first three higher-order solitons, but for a PCF with periodicity broken by the presence of a defect (absence of a hole). Note that in both cases the symmetry group is C_{6v} and that, in agreement with our previous result, the maximum vorticity allowed is two. The soliton solution with $l = 3$ is not a vortex. As predicted by group theory, it presents a binary phase structure (corresponding to a $|3;+-\rangle$ state) of the nodal-soliton type [8]. It is interesting to check the generality and accuracy of the group-theory approach using these numerical examples. The spectrum of higher-order soliton solutions is perfectly explained by our previous group-theory arguments, nevertheless the periodic (Fig. 4) and nonperiodic (Fig. 5) photonic crystal structures present notable differences. Despite that they share the same C_{6v} symmetry, a description in terms of weakly interacting localized fundamental solitons on lattice sites (the equivalent of the tight-binding approximation in solid state physics) [5] can only be valid in the perfectly periodic case. As is apparent in Fig. 4, this localization feature is clear in the amplitude and phase of vortex and nodal-soliton solutions in the periodic PCF. However, single fundamental solitons are no longer recognizable in the vortex and nodal solitons of Fig. 5 due to the presence of the periodicity-breaking defect. One can think of a situation of strongly interacting solitons causing the ''tight-binding approximation'' to stop being valid. Despite this fact, our main results concerning the nature of solutions and, more specifically, the restrictions on vorticity, remain valid with complete generality. On the other hand, these results pave the way for the manipulation of the vortex charge by means of external systems owning discrete symmetry. This singular property of discrete-symmetry systems can lead to applications in areas like optical data storage, distribution, and processing [1].

We are thankful to P. F. de Córdoba for useful discussions and to Grant Nos. TIC2002-04527-C02-02 (MCyT and FEDER funds), GV04B-390, and Grupos03/227 (Generalitat Valenciana).

[1] A. S. Desyatnikov, L. Torner, and Y. Kivshar, nlin.PS/0501026.
[2] G. A. Swartzlander and C. T. Law, Phys. Rev. Lett. **69**, 2503 (1992).
[3] A. W. Snyder, L. Poladian, and D. J. Mitchell, Opt. Lett. **17**, 789 (1992).
[4] J. Yang and Z. H. Musslimani, Opt. Lett. **28**, 2094 (2003).
[5] T. J. Alexander, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rev. Lett. **93**, 063901 (2004).
[6] Ferrando *et al.*, Opt. Express **12**, 817 (2004).
[7] Neshev *et al.*, Phys. Rev. Lett. **92**, 123903 (2004); Fleischer *et al.*, Phys. Rev. Lett. **92**, 123904 (2004).
[8] Ferrando *et al.*, Opt. Express **13**, 1072 (2005).
[9] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Addison-Wesley Series in Physics (Addison-Wesley, Reading, MA, 1964), 1st ed.
[10] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976).