Natural Velocity of Magnetic Islands

F. L. Waelbroeck
Institute for Fusion Studies, University of Texas, Austin, Texas 78712, USA
(Received 21 January 2005; published 12 July 2005)

The phase velocity of magnetic islands is calculated in the semicollisional regime with cold ions. Two solution branches arise, corresponding to islands propagating with the ions and with the electrons. For the ion branch the phase velocity and the polarization current are small. For the electron branch, the phase velocity depends on the ratio of W, the half-width of the island, and ρ_s, the Larmor radius calculated with the electron temperature. For $W \gg \rho_s$ the phase velocity is larger than the electron drift velocity and the polarization current is destabilizing. For $W \ll \rho_s$, the situation is reversed provided that the density and temperature gradients point in the same direction.

DOI: 10.1103/PhysRevLett.95.035002 PACS numbers: 52.30.Ex, 52.35.Py, 52.55.Tn

Magnetic islands are observed to affect the performance of stellarators [1], reversed-field pinches [2], and tokamaks [3]. It is thus important to understand the factors that determine their size. A key agent in the evolution of magnetic islands is the polarization current [4,5]. This current is caused by the distortion of the plasma flows under the influence of the magnetic island. Its effect depends on the difference between the velocity of the island and the velocity of the surrounding plasma, that is, on the phase velocity of the island. Interest in the possible stabilizing effect of the polarization current has been fueled by concerns regarding the effect of neoclassical tearing modes on future experiments [5,6].

An important consideration in determining the phase velocity is the degree of density flattening across the island [7,8]. If a density gradient is maintained inside the separatrix, the electrons experience a diamagnetic drift and the "frozen-in" property requires that the island's phase velocity match the electron drift velocity. If, by contrast, the density is flattened inside the island, the dominance of the ion viscosity compared to the electron viscosity causes the island to propagate at a velocity close to that of the ions [9,10]. Scott *et al.* [11] have shown that the density is flattened by sound waves when the amplitude of the mode is such that $k_\| c_s \gg \omega_*$, or equivalently $W \gg \rho_s L_s/L_n$, where $\omega_* = k\rho_s c_s/L_n$ is the diamagnetic frequency, $L_n = (\nabla \ln n)^{-1}$, k is the azimuthal wave vector, $k_\| = kW/L_s$ is the component of the wave vector along the magnetic field, L_s is the magnetic shear length, W is the island half-width, $\rho_s = c_s/\omega_{ci}$, $c_s = (T_e/m_i)^{1/2}$ is the sound speed, and ω_{ci} is the ion cyclotron frequency.

Recently, Ottaviani *et al.* [12] have reexamined the mechanisms responsible for density flattening, proposing that collisional parallel transport is the relevant mechanism for flattening when $k_\|^2 D_\| \gg k_\| c_s$, where $D_\| = T_e/0.51 m_e \nu_e$ and ν_e is the electron collision frequency. In this Letter, we restrict attention to the regime $W \ll \rho_s L_s/L_n$ where acoustic effects are unimportant, but we retain the parallel transport effects pointed out in Ref. [12]. We solve numerically and analytically the system of coupled equilibrium and transport equations derived previously by Connor *et al.* [13] and use the results to determine the phase velocity and the effect of the polarization current. The case of unsteady rotation has been examined by Mikhailovskii *et al.* [14].

The equilibrium and transport equations describe a class of steady-state solutions of the following two-fluid, cold-ion model consisting of Ohm's law and the vorticity, continuity, and electron heat equations, respectively:

$$D_t\psi + \nabla_\| n + \hat{\alpha}\nabla_\| T = Cj, \tag{1}$$

$$D_t U - \nabla_\| j = \mu \nabla_\perp^2 U, \tag{2}$$

$$D_t n - \nabla_\| j = D\nabla_\perp^2 n, \tag{3}$$

$$\frac{3}{2}D_t T - \hat{\alpha}\nabla_\| j = \frac{\hat{\kappa}}{C}\nabla_\|^2 T + \kappa_\perp \nabla_\perp^2 T. \tag{4}$$

We have normalized the time to ω_*^{-1}, the transverse distances (x) to ρ_s, and the azimuthal distances (y) to k^{-1}. Here $D_t = \partial/\partial t + \mathbf{v}_E \cdot \nabla$ where $\mathbf{v}_E = \hat{\mathbf{z}} \times \nabla\varphi$ is the electric drift, and $\nabla_\| = \hat{\mathbf{z}} \cdot \nabla\psi \times \nabla$ is the derivative in the direction of the magnetic field. The terms on the right hand sides of the equations represent transport phenomena, with $C = 0.51(\nu_e/\omega_*)(m_e/m_i)(L_s/L_n)^2$, $\hat{\alpha} = 1.71$, and $\hat{\kappa} = 1.09$. Lastly, $U = \nabla_\perp^2\varphi$ and $j = (\nabla^2\psi - 1)/\hat{\beta}$ describe the vorticity and the current, where $\hat{\beta} = \beta L_s^2/2L_n^2$ and β is the ratio of thermal to magnetic energy.

We are interested in the case of weak collisionality, $C \ll 1$, and assume that the other transport parameters are comparably small, so that the solutions depend only on the ratios of the transport parameters. We further assume that $\hat{\beta} \ll 1$. It follows then from Ampere's law, $\nabla^2\psi = 1 + O(\hat{\beta})$, that the constant-$\tilde{\psi}$ approximation applies so that $\psi = (1/2)[x^2 + w^2\sin^2(y/2)]$, where $w = W/\rho_s = (4\tilde{\psi}L_s/B_0)^{1/2}/\rho_s$ is the normalized half-width of the island. The half-width w is determined implicitly by the asymptotic matching condition

$$\hat{\Delta}' = \frac{16}{w^2}\int_0^\infty d\psi \langle j \cos y\rangle_\psi, \tag{5}$$

where $\hat{\Delta}' = \Delta' \rho_s/\hat{\beta}$ measures the drive for the tearing mode. The angular brackets $\langle\cdot\rangle_\psi$ represent the integral over a surface of constant flux, $\langle f\rangle_\psi = \oint dy f(x,y)/2\pi|x|$ for $\psi > w^2/2$. With the above approximations, the solution is determined by the single equilibrium parameter $\hat{\Delta}'$ as well as the three ratios of transport coefficients that control the profiles near the island.

The equilibrium and transport equations follow from the perturbative solution of Eqs. (1)–(4) in powers of the transport coefficients in a frame where the island is at rest [13]. To lowest order in the transport coefficients one finds $n = \varphi + H(\psi)$, and $j = -H'(\psi)(\varphi - \langle\varphi\rangle_\psi/\langle 1\rangle_\psi)$ where $H(\psi)$ is an integration constant. Eliminating the current between the continuity equation and the vorticity equation yields the ion continuity equation,

$$\hat{\mathbf{z}}\cdot[\nabla\varphi \times \nabla(n - U)] = O(C). \tag{6}$$

This equation has two distinct solutions. The first, $\varphi = O(C)$ implying $n = H(\psi) + O(C)$, describes an island at rest in the frame where the electric drift vanishes or, equivalently, an island propagating at the velocity of the ions. We refer to it as the ion branch. The ion branch occurs only when the island is sufficiently large to flatten the density inside the separatrix. With the cold-ion model used here the plasma flow is unperturbed for the ion branch and the polarization current vanishes. We henceforth restrict attention to the second solution branch, found by assuming that $\varphi = O(1)$ [13]. This is the relevant branch for thin islands.

The second solution of Eq. (6) yields an equilibrium equation for φ that is analogous to the Grad-Shafranov equation:

$$\nabla^2\varphi = K(\varphi) + H(\psi). \tag{7}$$

K and H are profile functions that are specified by the following two transport equations:

$$\frac{dH}{d\psi} = \frac{\hat{\kappa}D(1 - \langle x\varphi_x\rangle_\psi)\Gamma + \frac{3}{2}\hat{\alpha}\eta_e C\Upsilon}{\hat{\kappa}(\delta\langle x^2\rangle_\psi + C\Upsilon)\Gamma + \hat{\alpha}^2 C\Upsilon\langle x^2\rangle_\psi}, \tag{8}$$

where $\varphi_x = \partial\varphi/\partial x$, $\Gamma = \langle x^2\rangle_\psi + 9C\Upsilon/4\hat{\kappa}\kappa_\perp$, $\Upsilon = \langle\varphi^2\rangle_\psi - \langle\varphi\rangle_\psi^2/\langle 1\rangle_\psi$, and

$$\frac{dK}{d\varphi} = \frac{D}{\mu}\left(1 - \frac{1}{\bar{\varphi}_x}\right) + \sigma\left(\frac{D}{\mu} - 1\right)\frac{\overline{\partial_x H(\psi)}}{\bar{\varphi}_x}, \tag{9}$$

where the overline indicates an average along the streamlines. Equations (7)–(9), completed by the matching condition (5), constitute a complete set of nonlinear equations that determine the size and velocity of magnetic islands. We see that the solution of these equations describes an island propagating in the electron diamagnetic direction. We thus refer to their solution as the electron branch. Electron-branch solutions result when an island grown from the linear regime is insufficiently wide to flatten the density. The ion and electron branches described here are, respectively, consistent with the solution branches A and B recently discovered numerically by Ottaviani *et al.* [12]. The inclusion of acoustic effects in Ref. [12], however, leads to the excitation in type B solutions of a drift-acoustic wave that is absent from the model used here.

We restrict consideration to solutions such that φ is odd, so that $\varphi(0,y) = 0$ for all y. Because of the second-order nature of the equilibrium equation (7), it is necessary to supply an additional boundary condition for φ. We note that the change in the vorticity across the island region is proportional to the viscous force acting on the island: $\lim_{x\to\infty}[\langle U(x,y)\rangle - \langle U(-x,y)\rangle] = v_y'(\infty) - v_y'(-\infty) = F_y/\mu$, where F_y is the total viscous force. In the absence of an opposing electromagnetic force, F_y must vanish. Unfortunately, the boundary condition $U(x_e, y) = 0$ at *all* values of y, where x_e is the limit of integration, leads to boundary layers at the edge and convergence difficulties when solving the transport equations. We sidestep this problem by adopting the boundary condition $\varphi_x(x_e, y) = v_e$, where the edge velocity v_e is a free parameter. The natural velocity v_{free} is then equal to the value of v_e such that $\lim_{x\to\infty}\langle U(x,y)\rangle = 0$. Note that the phase velocity in the frame where the electric field vanishes is $v_{\text{phase}} = -v_{\text{free}}$. We next describe the numerical and analytic solution of the equilibrium and transport equations (8) and (9).

Numerical solution.—We have solved the equilibrium and transport equations numerically by a generalized Picard iteration procedure. The procedure makes use of the fact that the equilibrium equation (7) determines φ in terms of the profile functions K and H only, while the transport equations (8) and (9) determine K and H in terms of the electrostatic potential φ only. The algorithm thus consists of making an initial guess for the profile functions K and H and then solving the equilibrium and transport equations successively until the solution converges.

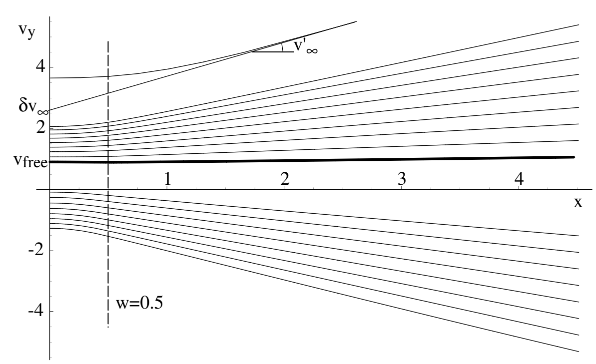

FIG. 1. Average along the streamlines of the azimuthal plasma velocity $\partial_x\varphi$ for various values of the applied torque. The parameters are $w = 0.5$, $\eta_e = 2$, $D = \mu = 0.6\kappa_\perp$, and $C = 1.65\kappa_\perp$ resulting in a natural propagation velocity $v_{\text{phase}} = -0.78$. Maintaining steady state requires $\hat{\Delta}' = 0.21$, indicating that the polarization current is stabilizing.

Figure 1 shows the average along the streamlines of the azimuthal velocity, $\bar{v}_y = \oint \varphi_x dy/2\pi$, for a scan of the edge velocity with $w = 0.5$. The solution corresponding to free propagation, $v'_\infty = 0$, is indicated by a thick line. The results show that the average of the central velocity deviates from the natural velocity by a small fraction of the difference between the edge and the natural velocities. Since the total electron velocity must vanish inside the separatrix, any variation in the electric drift inside the separatrix must be compensated by a corresponding change in the local diamagnetic drift. It follows that driving the flow in the direction of the electron drift velocity results in a *steepening* of the density profile inside the island.

The gap in Fig. 1 in the range $0 < \bar{v}(0) < v_{\mathrm{free}}$ corresponds to cases where the plasma velocity changes sign between the midplane and the edge. In such cases convergence fails, because the velocity reversal gives rise to convection cells. We note that the presence of convection cells for velocities intermediate between the background electric drift and the electron diamagnetic drift velocities is also predicted by solutions of the equilibrium equation that use a linear *ansatz* for $K(\varphi)$ [15].

The first two coefficients in the asymptotic form of the velocity at large x,

$$\bar{v}_y(x) = v'_\infty x + \delta v_\infty + O(1/x), \qquad (10)$$

provide a useful way to characterize the behavior of the velocity profile as the parameters vary. The geometric significance of these two coefficients is indicated in Fig. 1. The asymptotic shift in the velocity, δv_∞, measures the degree of permeability of the island to the plasma flow. The variation of δv_∞ as a function of $\delta v'_\infty$ for various island sizes is shown in Fig. 2. We find that thin islands ($w \ll 1$) are permeable, in the sense that δv_∞ is positive and approximately equal to the velocity of the plasma flow through the separatrix. For intermediate-sized islands ($1 \ll w \ll L_s/L_n$), by contrast, the velocity remains fixed at the natural velocity inside the separatrix, leading to $\delta v_\infty \simeq v_{\mathrm{free}} - v'_\infty w < 0$. We next describe analytic solutions of the equilibrium and transport equations.

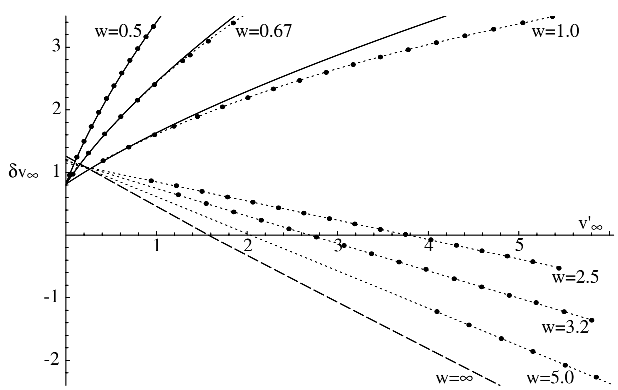

FIG. 2. Offset in the velocity as a function of the flow forcing for $D = \mu = 0.6\kappa_\perp$. $C = 1.65\kappa_\perp$ for the points with $w \leq 1$, while $C \ll \kappa_\perp$ for the points with $w > 1$. The solid and dashed lines represent the thin and intermediate width asymptotic solutions, and the dotted lines are quadratic fits.

Thin island regime $[(C\omega)^{1/2} \ll w \ll 1]$.—When the island width is such that parallel transport dominates over convective transport, $k_\parallel^2/C > \omega$, the island width replaces the semicollisional current channel as the scale of the thinnest current structure. We may solve the equilibrium equation in the limit $w \ll 1$ by observing that $\nabla^2 \sim w^{-2} \gg 1$, so that to lowest order, Eq. (7) reduces to Laplace's equation. The solution is $\varphi = v_0 x + O(w^2)$, where $v_0 = \delta v_\infty$ is the electric drift velocity close to the island. We determine v'_∞ by using the equilibrium equation in $v'_\infty = \lim_{x\to\infty} U = \lim_{x\to\infty}[K(v_0 x) + H(\psi)]$. Substituting the lowest-order solution $\varphi = v_0 x$ in the transport equations (8) and (9), we find that the asymptotic form of the profile functions at large x are $K(v_0 x) \sim (D/\mu) \times [(v_0 - 1)x + H(x^2)] - H(x^2)$ and $H(\psi) \sim H(x^2) \sim (1 - v_0)x + \delta H_\infty(v_0)$, where

$$\delta H_\infty(v_0) \equiv v_0 - 1 + \int_{w^2}^{\infty} d\psi \left(\frac{dH}{d\psi} - \frac{1 - v_0}{\psi^{1/2}}\right) \qquad (11)$$

represents the asymptotic shift in $H = n - \varphi$ with respect to its unperturbed value $(1 - v_0)x$. The asymptotic form of the permeability curves for thin islands is thus

$$v'_\infty = \delta H_\infty(\delta v_\infty) \qquad (12)$$

and the natural velocity is determined by solving $\delta H_\infty(v_{\mathrm{free}}) = 0$. Note that parallel force balance for the electrons imposes that H vanish inside the separatrix. This accounts for a contribution to the shift δH_∞ proportional to $(1 - v_0)w$. In the presence of a temperature gradient, however, there is an additional flattening outside the island due to the effect of the thermal force. This effect is represented by the term proportional to η_e in the expression for H' given in Eq. (8). The linearity of H' as a function of η_e makes it possible to separate the integral in Eq. (11) into a part proportional to η_e and a part multiplied by $1 - v_0$. We may thus use the condition for free propagation to write η_e as a ratio of two integrals that depend on the velocity. This defines the velocity implicitly in terms of η_e. For the parameters of Fig. 2, we find that the natural velocity is approximately given by $v_{\mathrm{free}} \simeq 1/(1 + 0.11\eta_e)$. This implies that the polarization drift is stabilizing [13]. Figure 2 compares the velocity calculated from (12) to that predicted by the direct solution of the equilibrium and transport equations. The agreement is surprisingly good for values of w as large as unity.

Moderate island regime $(1 \ll w \ll L_s/L_n)$.—We now consider the case of moderately large islands. In this case we may neglect the vorticity in the equilibrium equation (7). The lowest-order equilibrium solution is then $\varphi = K^{-1}[-H(\psi)] \equiv \Phi(\Psi)$ and $n = \Phi + H = n(\psi)$. The transport equations (8) and (9) can be seen to be degenerate in this limit. In order to find suitable transport equations we consider the flux-surface average of the vorticity equation

(2),

$$\mu\langle\nabla^2 U\rangle_\psi = \langle \mathbf{v}_E \cdot \nabla U\rangle_\psi = d/d\psi\langle U\nabla_\parallel \delta\varphi\rangle_\psi, \tag{13}$$

where $\delta\varphi$ is the small deviation from the equilibrium caused by transport fluxes. We may evaluate $\nabla_\parallel\delta\varphi$ by eliminating $\nabla_\parallel \delta n$ from Eqs. (2) and (3). Note that $j = O(w^{-2})$ so that the current can be neglected in (3). After eliminating $\nabla_\parallel \delta n$, we find $(M - L)\nabla_\parallel\delta\varphi = D\nabla^2 n$, where $M(\psi) \equiv d\Phi/d\psi$ and $L(\psi) = dn/d\psi$ are $\hat{\beta}^{-1/2}$ times the Alfvénic Mach numbers for the electric and diamagnetic drift velocities. Substituting this in the first integral of Eq. (13), we find the second-order transport equation

$$\frac{\mu}{D}\frac{d}{d\psi}\left(\langle x^4\rangle_\psi \frac{dM}{d\psi}\right) = \frac{L'M'}{M-L}\left(\langle x^4\rangle_\psi - \frac{\langle x^2\rangle_\psi^2}{\langle 1\rangle_\psi}\right). \tag{14}$$

The key question when considering the solution of Eq. (14) is that of the proper boundary condition at the separatrix, $\psi = \psi_s = w^2/2$. A natural choice is $M(\psi_s) = 0$, reflecting the idea that the velocity inside the separatrix should be vanishingly small for $W \gg \rho_s$ and that the velocity should be continuous across the separatrix. The solution of Eq. (14) in the absence of external forcing is then $M = 0$ everywhere, corresponding to the ion root mentioned earlier. This is clearly a valid solution, but the numerical results suggest that another, electron-branch solution exists such that the density gradient is maintained inside the island and the island propagates in the direction of the electron drift velocity.

In order to find a suitable boundary condition for the electron branch, we assume that the flow velocity takes a finite value $M_s = M(w^2/2)$ on the separatrix. The corresponding discontinuity at the separatrix is smoothed by finite ρ_s effects. We determine the value of M_s by solving the equilibrium and transport problems asymptotically in the region formed by joining the boundary layer lying outside the separatrix to the volume inside the separatrix. Expanding the right hand side of the equilibrium equation in this region, we obtain

$$\rho^2\partial_x^2\varphi - \varphi = -M_s x_s(y)[x - x_s(y)]\Theta[x - x_s(y)],$$

where $x_s(y) = w\cos(y/2)$ describes the position of the separatrix, Θ is the step function, and $\rho^2 = 1/K'(0)$. The solution of this equation is

$$\varphi = \frac{M_s z_s(y)}{K'(0)}\begin{cases}\sinh(z)e^{-z_s(y)}; & z < z_s(y),\\ \sinh(z_s(y))e^{-z} + [z - z_s(y)]; & z_s(y) < z,\end{cases}$$

where $z = x/\rho$. We may use this solution to evaluate the averages on the right side of the transport equations and thus to solve for $K'(0)$. Evaluating the stream averages on the midplane, $x = 0$, we find $\bar{\varphi}_x = 8M_s\rho^2/\pi$ and $\overline{\partial_x H} = 0$. Substituting this in Eq. (9) and solving for $K'(0)$ yields $K'(0) = 8/\pi w^2$. For such small values of $K'(0)$, the layer width ρ becomes large and the assumptions of the layer analysis are invalid. The point, however, is that the layer analysis shows that $K'(0)$ *cannot be finite*. Recalling that $\Phi = K^{-1}(H(\psi))$, we conclude that $H'(w^2/2) = -M_s K'(0) = O(w^{-2})$. Lastly, Eq. (9) implies that $\bar{\varphi}_x = 1$ so that $M_s = \langle x^2\rangle^{-1}_{\psi=\psi_s} = \pi/2$.

Figure 2 compares the solution of the vorticity transport equation (14) with $M_s = \pi/2$ to the direct solution of the equilibrium and transport equations. We find $v_{\text{free}} = 1.2$ corresponding to a destabilizing polarization drift, so that steady-state requires $\hat{\Delta}' = -1.02$.

In summary, we have investigated the propagation velocity of magnetic islands as a function of their size. The effect of the corresponding polarization current is important when the parameter $\hat{\beta} = \beta L_s^2/L_n^2 > \Delta'\rho_s$. When $W < \rho_s$ and $\eta_e > 0$, the electron branch propagates more slowly than the electron drift velocity, resulting in a stabilizing effect. When the island becomes larger than ρ_s, by contrast, the propagation velocity exceeds the electron drift velocity, resulting in a destabilizing effect. The density gradient inside the separatrix is maintained even for $W > \rho_s$, in agreement with the findings of Refs. [11,16]. Extrapolation of the present results to toroidal geometry and finite ion temperature leads one to conjecture that the critical island width for neoclassical island sustainment when $\Delta' < 0$ is the ion banana radius. Experiments show, however, that the critical width is typically 2 to 3 times larger than this [6]. This may be due to the stabilizing role of acoustic effects [10,12].

We are indebted to M. Ottaviani, F. Porcelli, and D. Escande for helpful discussions. This work was funded by U.S. DOE Contracts No. DE-FG03-96ER-54346 and No. DE-FC02-04ER54785.

[1] K. I. Ida *et al.*, Phys. Rev. Lett. **88**, 015002 (2002).
[2] B. E. Chapman *et al.*, Phys. Plasmas **11**, 2156 (2004).
[3] R. J. La Haye *et al.*, Phys. Plasmas **10**, 3644 (2003).
[4] A. I. Smolyakov, Plasma Phys. Controlled Fusion **35**, 657 (1993).
[5] H. R. Wilson *et al.*, Phys. Plasmas **3**, 248 (1996).
[6] O. Sauter *et al.*, Plasma Phys. Controlled Fusion **44**, 1999 (2002).
[7] D. Biskamp, Nucl. Fusion **19**, 777 (1979).
[8] L. E. Sugiyama and W. Park, Phys. Plasmas **7**, 4644 (2000).
[9] K. C. Shaing, Phys. Plasmas **10**, 4728 (2003).
[10] R. Fitzpatrick and F. L. Waelbroeck, Phys. Plasmas **12**, 22 307 (2005).
[11] B. D. Scott, A. B. Hassam, and J. F. Drake, Phys. Fluids **28**, 275 (1985).
[12] M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).
[13] J. W. Connor, F. L. Waelbroeck, and H. R. Wilson, Phys. Plasmas **8**, 2835 (2001).
[14] A. B. Mikhailovskii, S. V. Konovalov, and V. D. Pustovitov, Phys. Plasmas **7**, 2530 (2000).
[15] F. L. Waelbroeck, J. W. Connor, and H. R. Wilson, Phys. Rev. Lett. **87**, 215003 (2001).
[16] D. A. Monticello and R. B. White, Phys. Fluids **23**, 366 (1980).