P H Y S I C A L   R E V I E W   L E T T E R S

Equation of State and Collective Frequencies of a Trapped Fermi Gas Along the BEC-Unitarity Crossover

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We show that the study of the collective oscillations in a harmonic trap provides a very sensitive test of the equation of state of a Fermi gas near a Feshbach resonance. Using a scaling approach, whose high accuracy is proven by comparison with exact hydrodynamic solutions, the frequencies of the lowest compressional modes are calculated at $T = 0$ in terms of a dimensionless parameter characterizing the equation of state. The predictions for the collective frequencies, obtained from the equations of state of mean-field BCS theory and of recent Monte Carlo calculations, are discussed in detail.

The availability of Feshbach resonances and the consequent possibility of tuning the interatomic potential through the application of a magnetic field have recently stimulated much experimental and theoretical work on the crossover from Bose-Einstein condensation (BEC) to BCS in ultracold trapped Fermi gases [1]. Several relevant and already measured quantities, such as the release energy, the density profiles, and the frequency of the collective oscillations, are related in a sensitive way to the equation of state of the homogeneous system which varies significantly through the crossover. For small and negative values of the scattering length a the equation of state approaches the limit of a noninteracting Fermi gas, while for small and positive values, bound molecules can be formed and, at zero temperature, the system behaves as a dilute molecular Bose-Einstein condensed gas. In the intermediate regime important theoretical issues remain. For example, the intermolecular scattering length on the BEC side exhibits a nontrivial dependence on the free atom scattering length [2,3]. Furthermore, such interactions should give rise to beyond mean-field corrections in the equation of state [4], not accounted for by the Bogoliubov approximation. Similar corrections could also result from the composite nature of the molecules. Finally, near resonance, where the scattering length becomes larger than the average distance between particles, no simple many-body approach is applicable to this strongly correlated system. Numerical calculations of the equation of state of a uniform interacting Fermi gas at $T = 0$ along the crossover have been carried out in the past along the mean-field BCS (MF-BCS) approach [5] and, more recently, through *ab initio* simulations based on Monte Carlo (MC) algorithms [6,7].

Since the frequencies of the collective oscillations can be measured with high precision, it is of major interest to investigate their dependence on the equation of state along the crossover. From a careful and systematic analysis of the frequencies, one should be able to infer the actual equation of state. It has already been pointed out [8] that the collec-

tive frequencies of a $T = 0$ superfluid Fermi gas, trapped in a harmonic potential, approach well defined values in the important BEC and unitarity limit regimes, where the density dependence of the chemical potential can be inferred from general arguments. In the intermediate region, various investigations, based on the use of the hydrodynamic theory of superfluids and suitable parametrizations of the equation of state, have appeared recently [9–15]. In the mean time, first experimental results [16,17] on the frequencies of the lowest axial and radial compression modes on ultra cold gases of ^{6}Li across the Feshbach resonance have also become available.

The purpose of this Letter is to establish the accuracy needed in the measured frequencies in order to determine experimentally the equation of state. With this goal in mind, we will focus on two specific theories for the equation of state which exhibit a rather different density dependence of the chemical potential through the crossover. These are the MC calculations and the MF-BCS approximation discussed above. These equations of state are then employed within a hydrodynamic approach in order to calculate the relevant collective frequencies. Hydrodynamic theory is justified at $T = 0$ by the superfluid nature of the system and has been already successfully employed in trapped Bose-Einstein condensed gases. We will restrict here the discussion to the case of positive scattering lengths, where the experimental conditions for achieving the superfluid hydrodynamic regime are less severe, in contrast with the BCS regime where the smallness of the gap gives rise to stringent constraints [14]. The determination of the collective frequencies, for a given equation of state, is by no means a trivial task because, in general, the solutions must be found numerically in the nonsymmetric configurations relevant for experiments. Furthermore, since the frequencies vary only moderately with the scattering length, the analysis should be carried out with high precision. In the following, we will discuss only the lowest relevant modes, namely, the monopole

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mode for spherical geometry and the lowest axial compression mode for cigar configurations.

Let us first start discussing the two equations of state and, in particular, the dependence on density n of the chemical potential $\mu(n)$. Instead of the density, we use the Fermi wave vector k_F defined by $k_F^3 = 3\pi^2 n$. We restrict ourselves to the case of a wide Feshbach resonance where the scattering length a is the only relevant length associated with the two-body interaction, as it is indeed the case [18] for the dominant resonance in ^{6}Li and also in ^{40}K. This leads one to introduce the dimensionless coupling parameter $1/k_F a$, which varies from zero to infinity when one goes from unitarity to the bosonic limit of molecules. Similarly, we express the chemical potential in units of the Fermi energy $E_F = \hbar^2 k_F^2/2m$. The equations of state are displayed in Fig. 1.

At unitarity the result $\mu/E_F = 0.44$ extracted from MC calculations [6,7] is in fair agreement with experiments [19,20], while the MF-BCS theory gives a larger result $\mu/E_F = 0.59$. The slopes near unitarity are very similar, but the approach to the BEC limit is quite different, reflecting the fact that MC calculations are consistent with the recent prediction $a_M = 0.6a$ for the molecule-molecule scattering length [3], while the MF-BCS [5] yields the significantly larger value $a_M = 2a$.

In order to evaluate the frequencies of the collective modes, different routes have been followed in the literature. One class of methods assumes that the equation of state of the gas can be closely approximated by a polytropic expression $\mu(n) \simeq n^\gamma$ and makes use of the analytical results known [21] for this case. Note that the polytropic approximation is exact in the BEC regime where $\mu = gn$, as well as at unitarity where dimensionality considerations yield $\mu \propto n^{2/3}$. Other approaches are based on exact numerical solutions of the hydrodynamic equations [22]. Here we will consider another method
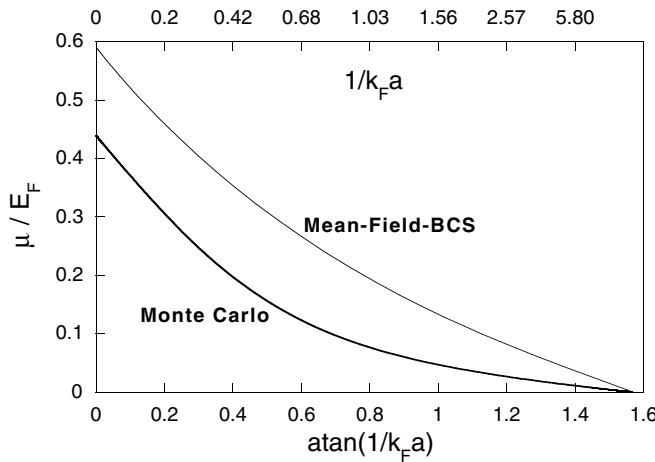

FIG. 1.   Chemical potential $\mu(n)$ (shifted by half the molecular binding energy $\hbar^2/2ma^2$) in units of the Fermi energy E_F as a function of the coupling parameter $1/k_F a$. The Monte Carlo data are extracted from Ref. [7].

based on the use of scaling transformations. Scaling has been already successfully employed to describe the dynamical behavior of Bose-Einstein condensates [23,24]. We will see that the results of the scaling approach agree remarkably well with exact solutions of hydrodynamic equations in all cases of practical interest we have considered. The advantage of the proposed method is its full applicability to anisotropic traps where exact numerical solutions are not always available. On the other hand, it can be also applied to regimes where the equation of state is not of the polytropic form.

In our scaling approach, we use the following ansatz for the space dependence of the density distribution:

$$n(x_1, x_2, x_3, t) = \gamma(t)n_0(a_1(t)x_1, a_2(t)x_2, a_3(t)x_3), \quad (1)$$

where $n_0(\mathbf{r})$ is the equilibrium density and $\gamma(t) = \Pi_i a_i(t)$. The factor $\gamma(t)$ ensures that the total particle number $\int d\mathbf{r}n(\mathbf{r}, t)$ is conserved. Correspondingly, we take $v_{s,i}(\mathbf{r}) = -(\dot{a}_i/a_i)x_i$ for the superfluid velocity field, which ensures that the equation of continuity $\dot{n}(\mathbf{r}) + \nabla[n(\mathbf{r})\mathbf{v}_s(\mathbf{r})] = 0$ is automatically satisfied. We will then insert the above ansatz for the density and for the velocity field into the hydrodynamic Lagrangian

$$\mathcal{L} = \int d\mathbf{r}\left[\frac{1}{2}mn(\mathbf{r})\mathbf{v}_s^2(\mathbf{r}) - e(n(\mathbf{r})) - n(\mathbf{r})V_{\text{ho}}(\mathbf{r})\right] \quad (2)$$

of our zero temperature superfluid. Here m is the atomic mass, and the chemical potential $\mu(n)$ is linked to the energy per unit volume $e(n)$ by $\mu(n) = \partial e(n)/\partial n$. The density dependent energy term $e(n)$ in the Lagrangian has been here obtained in the local density approximation, consistent with the hydrodynamic description.

In the following, we will specialize to the case of small oscillations, suited to the collective modes in the linear regime. Writing $a_i = 1 + \epsilon_i$, where $\epsilon_i \ll 1$, in the equations of motion obtained from the Lagrangian (2), we have to carry out the calculation up to first order in ϵ_i. After some algebra, we find the following simple eigenvalue equations for the mode frequencies ω:

$$\Gamma\omega_i^2\sum_j \epsilon_j + 2\omega_i^2\epsilon_i = \omega^2\epsilon_i, \quad (3)$$

where we have naturally used $\ddot{\epsilon}_i = -\omega^2\epsilon_i$ and we have defined the relevant dimensionless parameter

$$\Gamma = \frac{3}{2}\frac{\langle n\frac{\partial\mu}{\partial n}\rangle}{\langle V_{\text{ho}}\rangle} - 1, \quad (4)$$

where $V_{\text{ho}}(\mathbf{r}) = \Sigma_i(1/2)m\omega_i^2 x_i^2$ is the trapping harmonic potential with $x_1 = x$, $x_2 = y$, and $x_3 = z$. The brackets are for the equilibrium average over the gas cloud, for example, $\langle V_{\text{ho}}\rangle = \int d\mathbf{r}n_0(\mathbf{r})V_{\text{ho}}(\mathbf{r})$ and the density profile is evaluated in the Thomas-Fermi approximation $\mu(n) + V_{\text{ho}}(\mathbf{r}) = \mu_0$, consistent with the hydrodynamic description. Notice that the parameter Γ does not depend on the anisotropy of the trap. Using the virial theorem it is easy to prove that, for a polytropic equation of state $\mu(n) \propto n^\gamma$, the

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parameter Γ coincides with the exponent γ so that at unitarity one has $\Gamma = 2/3$, while in the BEC regime $\Gamma = 1$. In general, the coefficient Γ is a function of the dimensionless combination $k_F(0)a$, where $k_F(0)$ is fixed by the central value of the Thomas-Fermi density distribution according to $k_F^3(0) = 3\pi^2 n(0)$. The actual dependence of Γ on $k_F(0)a$ is determined only by the equation of state $\mu(n)$. Finally, the above derivation can be also reformulated as starting from an exact variational principle for the hydrodynamic frequencies.

In the isotropic case $\omega_i = \Omega$, in addition to the quadrupole mode $\omega^2 = 2\Omega^2$, Eq. (3) gives the result $\omega^2 = (3\Gamma + 2)\Omega^2$ for the monopole mode and, in particular, the well-known result $\omega = \sqrt{5}\Omega$ in the BEC limit $\Gamma = \gamma = 1$. In the axisymmetric geometry $\omega_1 = \omega_2 = \omega_\perp, \omega_1 = \omega_z$, we find, in addition to the quadrupole mode $\omega^2 = 2\omega_\perp^2$, the two $m = 0$ compressional solutions:

$$\omega^2 = \frac{1}{2}\left[2(\Gamma+1)\omega_\perp^2 + (\Gamma+2)\omega_z^2 \right.$$
$$\left. \pm \sqrt{[2(\Gamma+1)\omega_\perp^2 - (\Gamma+2)\omega_z^2]^2 + 8\Gamma^2\omega_\perp^2\omega_z^2} \right]. \quad (5)$$

For the relevant case of cigar geometry $\omega_z \ll \omega_\perp$, one finds $\omega^2 = \omega_z^2[3 - 1/(\Gamma + 1)]$ for the lowest axial compression mode, giving $\omega/\omega_z = \sqrt{5/2}$ in the BEC limit. For the radial compressional mode, one instead finds $\omega^2 = 2(\Gamma + 1)\omega_\perp^2$.

We will now show that the scaling result is extremely accurate by comparing its predictions with essentially exact solutions of the hydrodynamic equations. Our method [22] makes use of known exact analytical results for a large class of model equation of states $\mu_{anal}(n)$. Then the actual $\mu(n)$ is very closely approximated by one member of this class. Moreover, the effect of the small difference between $\mu(n)$ and $\mu_{anal}(n)$ is taken into account using a perturbative approach (this is called the "corrected" model). The absolute precision of this method has been checked [22] to be of order 10^{-3} at least. The relevant ingredients needed to calculate the mode frequencies are the reduced gas density in the trap $\bar{n}(\mathbf{r}) = n(\mathbf{r})/n_0$ and the corresponding chemical potential $\bar{\mu}(\mathbf{r}) = \mu(\mathbf{r})/\mu_0$, both normalized to their central values n_0 and μ_0. A class of soluble models is described by the law $\bar{n} = \bar{\mu}^p \exp[\Sigma_{k=0}^K P_k \bar{\mu}^k]$, where p and P_k are parameters. Here we consider only the cases $K = 0$ (which is the polytropic model $\bar{n} = \bar{\mu}^p$) and $K = 1$, because they give already by far a good enough precision. The model $K = 1$ is called "quasipolynomial." In this last case the solution is not fully analytical, but it involves only finding the proper root of a low order polynomial, which is a quite trivial numerical task [22].

We have carried out the calculations of the collective frequencies in two different physical situations. The first one is the isotropic geometry, which is particularly convenient for deriving exact solutions of the hydrodynamic equations. The second one is the elongated cigar geometry employed in many experimental setups. Note that the

results $\nu^2 = 4$ for the unitary case and $\nu^2 = 5$ for the BEC limit are much more separated in the spherical case than in the cigar geometry where one finds instead, for the lowest axial mode, $\nu^2 = 2.4$ and $\nu^2 = 2.5$, respectively.

We display our results in Fig. 2 for the spherical geometry. We have considered the two equations of state provided by MC calculations and by the MF-BCS approximation. The accuracy of the scaling approach is in both cases remarkable since at the scale of this figure it coincides with the exact result. The excellent quality of the scaling approach can be understood because it is exact not only for the polytropic case, but also if one includes first order corrections to the polytropic equation of state [25], as can be proven from the variational formulation mentioned above. In order to display the precision of our results, we have plotted the results in the inset of Fig. 2 on a much larger scale, taking the MC equation of state as an example. More precisely, we have taken as a reference the frequency ν^2_{qpcorr} obtained by the corrected quasipolynomial method, which we believe is the most accurate. Indeed, we see first that the corrected polytropic model and the corrected quasipolynomial are in remarkable agreement, since the difference between them is at most 8×10^{-4}. Hence the convergence of our method toward the exact result is almost already achieved at the level of the simpler corrected polytropic model. This clearly proves that the much more precise corrected quasipolynomial model is certainly extremely accurate, since it is without correction already within at most 10^{-2} of the exact result. The next striking and important result is the quite remarkable accuracy of the scaling method which is essentially never beyond 3×10^{-3} of the exact result for the MC equation of state (the case of the MF-BCS is even better). This gives a full validation to the scaling approach.
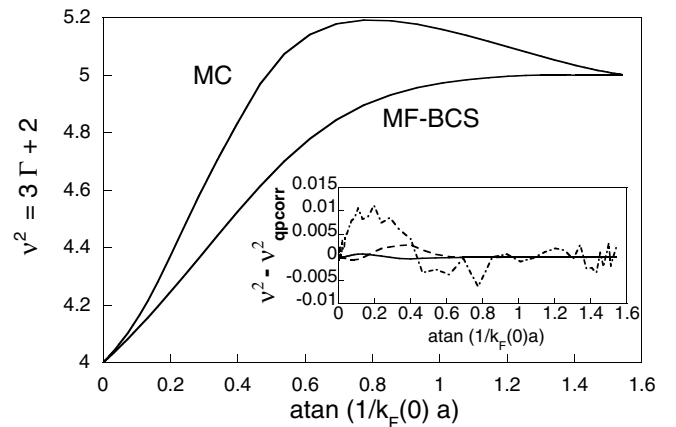

FIG. 2. Reduced square $\nu^2 = \omega^2/\Omega^2$ of the lowest monopole frequency in isotropic trap for the MC and the MF-BCS equation of states as a function of $1/k_F(0)a$, where $k_F(0)$ is the Fermi wave vector at the trap center. Solid lines correspond to the results of all our methods, which cannot be distinguished at this scale. Inset: (uncorrected) quasipolynomial (dot-dashed line), corrected polytropic (solid line), and scaling (dashed line), all compared to the corrected quasipolynomial model.
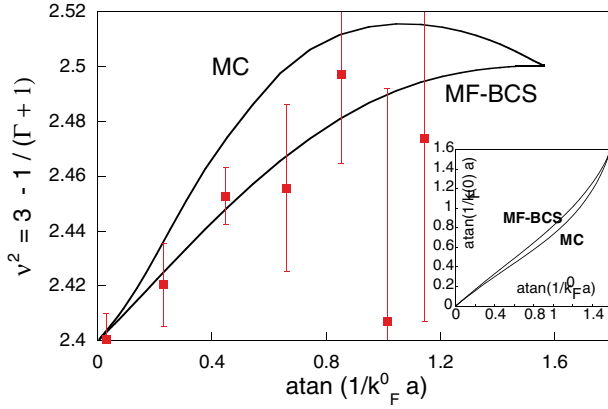
FIG. 3 (color online). Reduced square $\nu^2 = \omega^2/\omega_z^2$ of the axial mode frequency in very elongated trap for the MC and the MF-BCS equation of states, as a function of $k_F^0 a$, where $k_F^0 = 1.7 N^{1/6}/a_{\mathrm{ho}}$ is the Fermi wave vector of the ideal Fermi gas in a harmonic trap and $a_{\mathrm{ho}} = (\hbar/m\bar{\omega})^{1/2}$, where $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$. Experimental results are from Ref. [17]. The inset gives the relation between k_F^0 and $k_F(0)$, used in Fig. 2.

An essential feature emerging from the results of Fig. 2 is that the separation between the MC and the MF-BCS results is definitely much larger than any theoretical uncertainty within the $T = 0$ hydrodynamic picture. It should be consequently possible to distinguish them experimentally. In particular, the observation of an enhancement of ν^2 above 2.5, for $k_F^0 a \sim 1$, would provide evidence for beyond mean-field corrections [4,26], which are absent in MF-BCS.

Finally, we turn to the cigar geometry. We have carried out an analysis of the lowest axial compressional mode by comparing the results obtained from the scaling method with our exact results, in the case studied in Ref. [15]. Again the agreement is perfect. In Fig. 3 we report the predictions of the scaling approach applied to the MC and the MF-BCS equations of state. In the same figure, we also show the experimental results from [17] for the axial mode [27]. It is quite intriguing that these data look in better agreement with the mean-field BCS equation of state than with the Monte Carlo one. This might be due to finite temperature effects. Clearly further experimental work would be important to settle this point, which is essential for our understanding of the BEC-BCS crossover, and, in particular, to decide which equation of state better describes the experimental reality.

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