Quantum State Reconstruction via Continuous Measurement

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We present a new procedure for quantum state reconstruction based on weak continuous measurement of an ensemble average. By applying controlled evolution to the initial state, new information is continually mapped onto the measured observable. A Bayesian filter is then used to update the state estimate in accordance with the measurement record. This generalizes the standard paradigm for quantum tomography based on strong, destructive measurements on separate ensembles. This approach to state estimation induces minimal perturbation of the measured system, giving information about observables whose evolution cannot be described classically in real time and opening the door to new types of quantum feedback control.

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The control of quantum mechanical systems is finding new applications in information processing tasks such as cryptography and computation [1]. Experimental reconstruction of a quantum state is essential to verify preparation, to detect the presence of errors due to noise and decoherence, and to determine the fidelity of control protocols using process tomography. Moreover, real-time "state estimation" may allow improvement of precision metrology beyond the standard quantum limit [2], with the possibility of active control through closed-loop feedback protocols [3]. In addition, measurement of the quantum state can provide information about unknown or nontrivial dynamics, such as those arising in the study of quantum chaos. Laboratory demonstrations of state reconstruction are numerous and span a broad range of physical systems, including light fields [4], molecules [5], ions [6], atoms [7], spins [8,9], and entangled photon pairs [10].

In this Letter we consider a new protocol for quantum state reconstruction based on continuous, weak measurement of a single observable on a single ensemble of identically prepared systems. The ensemble is driven so that each member undergoes an identical, carefully designed dynamical evolution that continually maps new information onto the measured quantity. This is in contrast to the standard paradigm for quantum state reconstruction based on strong measurements, often of a large set of observables performed on many copies of the unknown state. Our weak measurement approach has a number of possible advantages in situations that lend themselves naturally to working with ensembles. Strong measurements on ensembles are inefficient because only a single observable can be measured after each preparation and the information gained about the observable is extracted independent of the required fidelity. By contrast, weak measurements can be optimized to obtain just enough information to estimate the density matrix to some required fidelity, in real time, and with minimal disturbance of each member. The extracted information could be used to perform closed-loop feedback control based on knowledge about the entire quantum state extending protocols based on state estimation of Gaussian random variables that evolve according to classical dynamics [11]. Our procedure is broadly applicable in systems where continuous weak measurement tools have been developed, such as nuclear magnetic resonance in molecules [8] and polarization spectroscopy in dilute atomic vapors [12], but where noise and decoherence limits the ability to perform strong measurements regardless of the amount of signal averaging.

In our protocol we consider an ensemble of identically prepared systems, $\rho_N = \rho_0^{\otimes N}$, whose dynamical evolution is driven in a known fashion and monitored by a probe that measures the sum of the identical observables $\{O\}$ on each member. Because of the central limit theorem the measurement record of this probe has the form

$$M(t) = N\langle O\rangle_t + \Delta M(t), \tag{1}$$

where $\langle O\rangle_t$ is the quantum expectation value at time t, and $\Delta M(t)$ is a Gaussian white noise process with variance $\sigma^2 = 1/\kappa\Delta t$ for measurement strength κ and detector averaging time Δt. In principle, a measurement of the collective observable $N\langle O\rangle_t$ leads to backaction on the collective many-body state and can cause individual members of the ensemble to become correlated [3,13]. Such correlations influence the outcome of future measurements and greatly complicate the task of reconstructing the initial state ρ_0. Additionally, the gain from performing such quantum limited measurements is small, as the majority of the information about the state of individual ensemble members has already been extracted by the probe prior to reaching the quantum limited regime. We thus restrict our considerations to cases where the measurement uncertainty, averaged over the total measurement time T, is large compared to the intrinsic quantum uncertainty (projection noise) of the collective observable, $1/\kappa T > N\Delta O^2$, and backaction onto the collective state is insignificant. Experimentally, this is also the most common situation.

A sufficient measurement signal-to-noise ratio (SNR) must still be available to reconstruct the state of an individual member of the ensemble. This requires $N \gg 1$ so that the quantum backaction associated with information gain is distributed uniformly among the entire ensemble, with negligible disturbance of any single member state.

The goal is to invert the measurement history, Eq. (1), to determine ρ_0. As we wish this procedure to be independent of ρ_0, it is most convenient to work in the Heisenberg picture and express $\langle O\rangle_t = \mathrm{Tr}(O(t)\rho_0) = (O(t)|\rho_0)$, where in the second equality we have written the trace as an inner product between "superoperators" [14]. We coarse grain over the detector response time Δt, such that $(O_i| = \int_t^{t+dt}(O(t)|dt/\Delta t$, obtaining a discrete measurement history time series $\{M_i\}$, with $M_i = N(O_i|\rho_0) + \sigma W$, where now the measurement operators $\{O_i\}$ are determined in advance by the known dynamics, and where W is a Gaussian random variable with zero mean and unit variance. This equation recasts the reconstruction problem as a stochastic linear estimation problem for the underlying state ρ_0.

In order to reconstruct the state from the measurement time series, the set of measurements operators $\{O_i\}$ must span the space of density operators, i.e., the dynamics must map the initial measurement to all possible (Hermitian) measurements. This is best achieved by introducing an explicit set of control parameters, with a time-dependent series of Hamiltonians $\{H_i\}$. We require that this set generate the Lie algebra for $SU(d)$, where d is the dimension of the space; the system must be "controllable". Under that assumption, and in the absence of decoherence, weak measurement on an infinitely large ensemble can be used to achieve arbitrary fidelity in the reconstruction by increasing the averaging time, thereby reducing the uncertainty. In the presence of decoherence, however, one must apply a well chosen time-dependent Hamiltonian series that accesses the requisite observables with sufficient fidelity before the state decays.

Including the coupling to the environment, the evolving measurement operators can then be expressed in terms of the base observable as $(O(t)| = (O|S_t$ where $S_t = \mathbb{T}\exp[\int_0^t dt' \mathcal{L}_{t'}]$ with $\mathcal{L}_t$ the generator of the dynamics and $\mathbb{T}$ the time ordering operator. To simulate this evolution we consider a time scale δt over which $\mathcal{L}_t$ changes negligibly. The semigroup property then allows us to approximate $S_{t+\delta t} = e^{i\mathcal{L}_t \delta t} S_t$ which can be numerically calculated given a system of reasonable size, $d < 100$.

A Bayesian filter determines how our knowledge of ρ_0 is updated due to a measurement history $\{M_i\}$,

$$P(\rho_0|\{M_i\}) = AP(\{M_i\}|\rho_0)P(\rho_0). \tag{2}$$

Here A is the normalization constant for the posterior distribution and $P(\rho_0)$ contains the prior information, including the fact that ρ_0 is a valid positive trace-one operator. $P(\{M_i\}|\rho_0)$ is the conditional distribution that contains the information gained during the experiment. The conditional distribution quantifies how well the measurement performs. Because of the Gaussian measurement statistics, this distribution has the form,

$$P(\{M_i\}|\rho_0) = \prod_i C \exp\left\{-\frac{[M_i - N(O_i|\rho_0)]^2}{2\sigma^2}\right\} \propto \exp[-(\delta\rho|\mathcal{R}|\delta\rho)]. \tag{3}$$

The superoperator $\mathcal{R}$ is the covariance matrix of the multidimensional Gaussian and $\delta\rho = \rho_0 - \hat{\rho}$ is the difference between the prepared state and the maximum likelihood estimate of this state given the measurements, equivalent to the least squares estimator for a Gaussian random variable. Following Eq. (3),

$$\mathcal{R} = \frac{N^2}{\sigma^2}\sum_i |O_i)(O_i|, \quad |\hat{\rho}) = \frac{N}{\sigma^2}\sum_i M_i \mathcal{R}^{-1}|O_i). \tag{4}$$

This evolving covariance matrix generalizes the classical update rule discussed in [11]. Full state reconstruction requires information about all $d^2 - 1$ independent operators which implies the need for higher order moments of the base observable if the system is larger than spin 1/2.

If the covariance matrix is full rank then the measurement history is informationally complete with the $d^2 - 1$ primary uncertainties given by the eigenvalues of the covariance matrix. Specifically, the conditional probability distribution has entropy

$$S = -\frac{1}{2}\log\mathcal{R} = -\sum_j \log\sqrt{\lambda_j}, \tag{5}$$

where λ_j are the eigenvalues of the covariance matrix, corresponding to the inverse of the variances of Eq. (3) along its primary axes. This entropy provides a collective measure of the information gained about all parameters, independent of the initial state and any prior information. To obtain an accurate reconstruction we need to optimize the entropy and any additional costs over the free parameters (controls).

Given the measured information the maximum likelihood estimate of the quantum state is the mean of the Gaussian conditional distribution, as given by the least squares fit, Eq. (4). This, however, does not take into account the prior information that $\mathrm{Tr}(\rho_0) = 1$ and positivity. The trace condition is ensured by adding a pseudomeasurement of the trace $M_0 = I/d$, which has zero variance. To enforce positivity, one could solve for the closest positive state to $\hat{\rho}$ using convex optimization [15]. Alternatively, one can get a reasonable and much simpler estimate by setting the negative eigenvalues of $\hat{\rho}$ to zero, and renormalizing to give $\hat{\rho}_{\mathrm{pos}}$. This simpler procedure is used in the example below as the method employed to enforce positivity has only a weak effect on reconstruction performance in general, though it may have a strong effect for a few specific states.

As a concrete demonstration of the power and versatility of our method we consider the reconstruction of the quantum state associated with the total spin-angular momentum of an ensemble of alkali atoms, in our specific example the $F = 3$ or $F = 4$ hyperfine manifolds of the $6S_{1/2}$ ground state of ^{133}Cs. The number of parameters needed for reconstruction are then $(2F + 1)^2 - 1$, giving 48 and 80 components, respectively. Consider a cloud of atoms prepared in identical states ρ_0, and coupled to a common, linearly polarized, probe beam tuned near the D1 ($6S_{1/2} \rightarrow P_{1/2}$) or D2 ($6S_{1/2} \rightarrow P_{3/2}$) resonance [12]. Information about the atomic spins is obtained by measuring the Faraday rotation of the probe polarization, which provides a continuous measurement of the spin component along the direction of propagation, $O = F_z$. Shot noise in the probe polarimeter gives rise to the fluctuations W which limit the measurement accuracy.

In the regime of strong backaction onto the collective spin state, such measurements have been used to generate spin squeezed states [3,13] and to perform subshot noise magnetometry [3,11]. In the regime of negligible backaction that is of interest here, Smith *et al.* continuously monitored the Larmor precession of spins in an external magnetic field and observed a series of dynamical collapses and revivals due to a nonlinear term in the spin Hamiltonian [16] resulting from the ac Stark shift caused by off-resonance excitation. This nonlinearity allows for full controllability of the atomic spin and reconstruction of the input quantum state. Off-resonance excitation also introduces a small but unavoidable amount of decoherence due to photon scattering. Quantum state reconstruction requires a large enough nonlinearity to generate dynamics that cover the entire operator space before decoherence erases information about the initial state. To generate this with the ac Stark shift, one requires large excited state hyperfine splittings.

To control the system we apply a time-dependent magnetic field. The overall Hamiltonian, including the nonlinear ac Stark shift induced by an x-polarized probe, is [16]

$$H(t) = g_F \mu_B [\mathbf{B}(t) + \mathbf{B}_0] \cdot \mathbf{F} + \beta \hbar \gamma_s F_x^2, \qquad (6)$$

assuming that multiple scattering inside the ensemble is negligible. $\mathbf{B}(t)$ is the control field and $\mathbf{B}_0$ represents any background field that might be present in an experiment. In the nonlinear term, we have factored out the scattering rate γ_s for a transition with unit oscillator strength, and introduced the detuning dependent ratio β between the time scales for coherent evolution and decoherence due to optical pumping. In this example we explore two regimes: a probe detuned from the D2 transition by much more than the excited state hyperfine splitting, in which case beta is approximately independent of detuning, ($\beta = 0.81$), and a probe tuned halfway between the two excited hyperfine states ($\beta = 7.67$). The evolution of the ensemble is governed by the master equation $\mathcal{L}_t[\rho] = -\frac{i}{\hbar} \times [H(t), \rho] - \frac{\gamma_s}{2}\mathcal{D}[\rho]$, where ρ has support only on the ground state of interest and all other states have been adiabatically eliminated. Optical pumping within and out of the initial hyperfine manifold is accounted for by $\mathcal{D}[\rho]$. To simulate this evolution we construct the $\mathcal{L}_t$ for some choice of scattering rate, background field, and controls $\mathbf{B}(t)$. The measurement strength κ is determined empirically by the shot-noise limited measurement uncertainty σ, which we characterize by the signal-to-noise-ratio SNR $= M_{\max}/\sigma$, where $M_{\max} = \max_\rho (F_z|\rho)$ is the maximum signal possible. We account for inhomogeneous values of $\mathbf{B}_0$ by averaging over a Gaussian distribution corresponding to a standard deviation of 60 Hz in the induced Larmor frequency. The duration of the simulated measurement is $T = 4$ ms, the coarse graining time is 4 μs, and the average photon scattering rate is $\gamma_s = 10^3\ \mathrm{s}^{-1}$. Finally, we simulate the effect of a low pass filter by averaging our measurement over a few coarse graining time steps.

We optimize the controls based on a cost function consisting of the entropy $S(\mathcal{R})$ Eq. (5) and any additional control costs $C(\mathbf{B})$. In this case the additional costs include the degradation in reconstruction due to loss of field control at large amplitudes, and inability to rapidly change the fields. We minimize these costs over all possible time-dependent magnetic fields $\mathbf{B}(t)$. This is done sequentially, first restricting the magnetic fields to optimize the control costs $C(\mathbf{B})$, and then optimizing the entropy subject to these restrictions. The magnetic field is restricted to be in the x-y plane, with magnitude $|\mathbf{B}| = B$, such that the Larmor frequency is $\omega_B = \mu_B B/h = 15$ kHz. Additionally, we specify the field at only $n = 50$ independent times and smoothly interpolate between them to ensure slow variation. The only free parameters are thus a set of $n = 50$ independent angles. The entropy optimization landscape subject to these constraints has many local minima precluding the use of purely local search techniques. Instead, we use a one dimensional global search, where we iteratively optimize one of the $n = 50$ independent angles, holding the others fixed. This process is repeated until all of the angles are independently globally stationary.

The results of simulated reconstructions are shown in Figs. 1 and 2. Given an initial preparation in the $F = 3$ hyperfine manifold, in the "cat state" $|\psi_0\rangle = (|m = 3\rangle + |m = -3\rangle)/\sqrt{2}$, the fidelity of the reconstruction, $\mathcal{F} = \langle\psi_0|\hat{\rho}_{\rm pos}|\psi_0\rangle$, averaged over 1000 noise realizations, is plotted versus simulated SNR. Increasing SNR clearly results in a better reconstruction. The parameters needed to attain good fidelity for reconstruction of the $F = 3$ ground state using the D1 transition (Fig. 1) appear to be well within the reach of current experiments at $\mathcal{F} \approx .95$, SNR $= 30$ [12]. Even with a possible 1% uncertainty in the control fields, a fidelity of $\mathcal{F} = .85$ should be possible

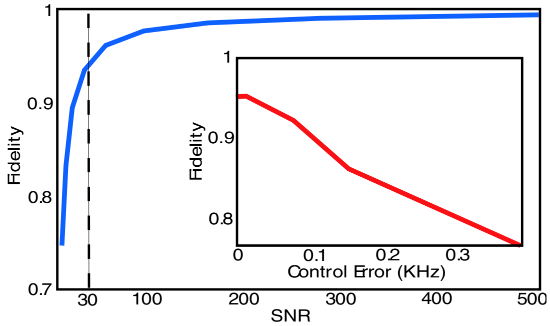

FIG. 1 (color online). Quantum state reconstruction of the cat state described in the text for the $F = 3$ hyperfine ground state of ^{133}Cs probed on the D1 transition. The figure shows the fidelity of reconstruction as a function of the measurement signal to noise ratio. Experimentally, an SNR of 30 is readily accessible, and SNRs of up to 100 have been achieved in a single run. In this example the large nonlinearity allows good fidelity reconstruction based on a single trajectory through state space, as indicated by the dashed line where a fidelity $\mathcal{F} = 0.95$ is reached at an SNR of 30. The inset shows how the reconstruction degrades due to errors in the control field at a constant SNR of 30. For a reasonable 1% control field error the fidelity drops to $\mathcal{F} = 0.85$.

(Fig. 1 inset). A different regime is illustrated in the reconstruction of a spin $F = 4$ state using the D2 resonance. Here we attempt to reconstruct more parameters, 80 versus 48, and use a smaller nonlinearity because of the smaller hyperfine splitting of the $P_{3/2}$ manifold. Reconstruction here is infeasible with a single measurement run performed on a single ensemble, even assuming very large SNR (Fig. 2). High fidelity reconstruction is obtained only when we combine the measurement records from multiple independent runs, each starting with a fresh ensemble, and explore operator space using distinct trajectories.

We have presented a new protocol for quantum state reconstruction based on continuous measurement of an ensemble of N members, and demonstrated our procedure through a simulated reconstruction of a spin J via polarization spectroscopy of a gas of cold atoms. The reconstruction technique is nondestructive and exploits classical estimation theory, providing a starting point for consideration of more complex applications of quantum control tasks such as feedback. In future work we plan to improve our optimization procedure for robustness in control-parameter uncertainty and examine global search procedures such as convex optimization [15]. The tools developed here should provide new avenues for real-time state quantum estimation that allow us to explore the dynamical generation of nonclassical features, such as entanglement. This is of particular interest for mesoscopic systems whose classical description exhibits chaos [17].

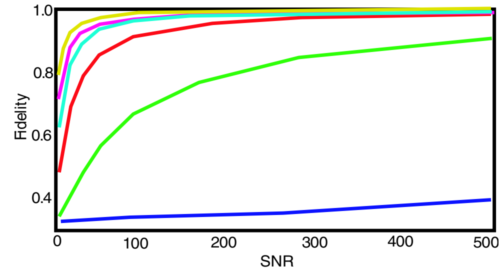

FIG. 2 (color online). Quantum state reconstruction for the $F = 4$ hyperfine state probed on the D2 transition, where the nonlinearity is weaker than for the case in Fig. 1. The curves correspond to an increasing number of distinct trajectories explored in independent realizations of the experiment, which allows increasingly complete coverage of the operator space. From lowest to highest fidelity we use: 1 run (blue), 2 runs (green), 3 runs (red), 4 runs (cyan), 5 runs (magenta), and 6 runs (yellow). Here, even at an unreasonable SNR of 500, reconstruction is not possible in a single run, though similar performance to that achieved in Fig. 1 can be obtained using 4–5 independent runs.

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[1] M A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[2] JM Geremia *et al.*, Phys. Rev. Lett. **91**, 250801 (2003).
[3] JM Geremia, J. K. Stockton, and H. Mabuchi, Science **304**, 270 (2004); JM Geremia, J. K. Stockton, and H. Mabuchi, Phys. Rev. Lett. **94**, 203002 (2005).
[4] D. T. Smithey *et al.*, Phys. Rev. Lett. **70**, 1244 (1993).
[5] T. J. Dunn, I. A. Walmsley, and S. Mukamel, Phys. Rev. Lett. **74**, 884 (1995).
[6] D. Leibfried *et al.*, Phys. Rev. Lett. **77**, 4281 (1996).
[7] C. Kurtsiefer, T. Pfau, and J. Mlynek, Nature (London) **386**, 150 (1997).
[8] I. L. Chuang, N. Gershenfeld, and M. Kubinec, Phys. Rev. Lett. **80**, 3408 (1998).
[9] G. Klose, G. Smith, and P. S. Jessen, Phys. Rev. Lett. **86**, 4721 (2001).
[10] A. G. White *et al.*, Phys. Rev. Lett. **83**, 3103 (1999).
[11] K. Mølmer and L. B. Madsen, Phys. Rev. A **70**, 052102 (2004).
[12] G A. Smith, S. Chaudhury, and P. S. Jessen, J. Opt. B **5**, 323 (2003).
[13] A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. **85**, 1594 (2000).
[14] C M. Caves, J. Supercond. **12**, 707 (1999).
[15] L. Vandenberghe and S. Boyd, SIAM Rev. **38**, 49 (1996).
[16] G. A. Smith *et al.*, Phys. Rev. Lett. **93**, 163602 (2004).
[17] A. Lakshminarayan, Phys. Rev. E **64**, 036207 (2001).