Flow Reversal in a Simple Dynamical Model of Turbulence

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In this Letter, we study a simple hydrodynamical model showing abrupt flow reversals at random times. For a suitable range of parameters, we show that the dynamics of flow reversal is accurately described by stochastic differential equations, where the noise represents the effect of turbulence.

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It has been recently reported [1] that abrupt flow reversal takes place at a large Rayleigh number in thermal convection. In addition to thermal convection, flow reversal has been also observed in laboratory experiment of two dimensional turbulence [2] and in the magnetic polarity of Earth [3]. More generally, there are many "turbulent" flows for which transitions between different states have been investigated, namely, within the theory of multiple equilibria for atmospheric flows [4] and of long time climatic changes [5]. In most cases the major question to be answered by experiments or observations concerns the mechanism responsible for the transition. This question is highly nontrivial whenever the average persistent time $\langle \tau \rangle$ around different states is much longer than any characteristic times describing the dynamical behavior of the system.

There are two main interpretations which have been suggested so far and pointed out in Refs. [1–6]. The first one assumes that turbulence is more or less a "noise" applied to the order parameter ψ which describes the system (i.e., the wind in the case of Rayleigh-Benard convection or the temperature in the case of climatic change). Then, by defining ψ such that the two observed states are $\pm\psi_0$, the equation for ψ is given by

$$d\psi = \left[a\psi\left(1 - \frac{\psi^2}{\psi_0^2}\right)\right]dt + \sqrt{\sigma}dW(t), \tag{1}$$

where W is an incremental Wiener process δ correlated in time with zero mean and variance dt, while a^{-1} is the characteristic time scale of the instability at $\psi = 0$. One can show that transitions between the two stable states $\pm\psi_0$ occur at random times τ with an average time $\langle \tau \rangle$ given by

$$\langle \tau \rangle \sim \frac{\pi}{\sqrt{2}a} \exp[a\psi_0^2/(2\sigma)] \tag{2}$$

Hereafter, following the language of stochastic differential equations, the random time τ will be referred to as "exit time." Note that Eq. (1) implies, for small σ, that the average exit time $\langle \tau \rangle$ is much longer than the deterministic "fluidodynamical" time a^{-1}. Moreover, by employing the theory of stochastic differential equations [7], one can compute the probability distribution of the exit time τ, which, for small σ, is given by

$$P(\tau) = \langle \tau \rangle^{-1} \exp(-\tau/\langle \tau \rangle). \tag{3}$$

If the above scenario is believed to be correct, then the transitions between the two states $\pm\psi_0$ are due to repeated small noise perturbations of the same "sign" which are acting against the deterministic "force"; i.e., there is no specific mechanism introduced by the small scale turbulence (parametrized by the noise) and transitions can be explained in terms of large deviation theory.

One of the major criticisms against the above interpretation is that the noise *is* by itself the effect of turbulence and, in most cases, there is no time scale separation between the dynamic fluctuations of ψ and turbulent fluctuations. Thus one cannot assume that "the noise" is a "fast" perturbation with respect to the dynamics of ψ and, as a consequence, Eq. (1) cannot be justified. As an alternative, one should look for a specific fluidodynamical large scale mechanism which can explain the observed transitions. For instance, for the wind reversal in thermal convection, there has been a recent proposal by [6] which explains transitions as the result of plume dynamics.

In this Letter we want to understand whether and how Eq. (1) can be justified, at least in the simplest possible model of a turbulent flow. For this purpose we shall consider an "energy cascade" model, i.e., a shell model aimed at reproducing few of the relevant characteristic features of the statistical properties of the Navier-Stokes equations [8,9]. In a shell models, the basic variables describing the "velocity field" at scale $r_n = 2^{-n}r_0 \equiv k_n^{-1}$, is a complex number u_n satisfying a suitable set of nonlinear equations. There are many version of shell models which have been introduced in literature. Here we choose the one referred to as the *Sabra* [10] shell model:

$$\frac{du_n}{dt} = ik_n[a\Lambda u_{n+1}^* u_{n+2} + bu_{n-1}^* u_{n+1} - c\Lambda^{-1} u_{n-2}u_{n-1}] - \nu k_n^2 u_n + f_n \tag{4}$$

where $\Lambda = 2$, $a = 1$, and $c = (1 + b)$ and f_n is an external forcing. Let us remark that the statistical properties of intermittent fluctuations, computed either using u_n or the instantaneous rate of energy dissipation, are in close *qualitative* and *quantitative* agreement with those measured in laboratory experiments, for homogeneous and isotropic turbulence [10].

The basic idea of our approach is to assume that $U_r \equiv \mathrm{real}(u_1)$ describes the one dimensional unstable manifold arising by a (large scale) pitchfork bifurcation. Consequently we change the equation for u_1 as follows:

$$\frac{du_1}{dt} = \Phi + \mu u_1\left(1 - \frac{u_1^2}{u_0^2}\right) - \nu k_1^2 u_1, \tag{5}$$

$$\frac{du_n}{dt} = ik_n[a\Lambda u_{n+1}^* u_{n+2} + bu_{n-1}^* u_{n+1} - c\Lambda^{-1}u_{n-2}u_{n-1}] - \nu k_n^2 u_n.(n > 1), \tag{6}$$

where $\Phi \equiv ik_1 a\Lambda u_2 u_3^*$. Let us comment on Eq. (5). In most cases, a pitchfork bifurcation, as described by Eq. (5), is observed in real fluidodyanamical flows with respect to the external forcing or the Reynolds number. In our case we are assuming that the unstable manifold is coupled to smaller scales by the term $ik_1 a\Lambda u_2^* u_3$. For small ν, the two states $u_1 = \pm u_0$ become unstable and a turbulent regime is observed. In the following we will think of Eq. (5) as a realistic, although approximate, equation describing a simplified turbulent "flow" superimposed to a large scale instability. As one can see, no *external noise* is introduced in the system. We remark that Eqs. (5) and (6) are invariant under the transformation (u_{3m+1}, u_{3m+2}, u_{3m+3}) with ($-u_{3m+1}$, $-u_{3m+2}$, u_{3m+3}) ($m = 0, 1, 2, \ldots$), which implies that the probability distribution for u_1 is symmetric. Using dimensionless variables $W_n = u_n/u_0$, $K_n = k_n L$, and $t' = \mu t$ ($L \equiv k_1^{-1}$), one gets

$$\frac{dW_1}{dt'} = i\frac{u_0}{\mu L}K_1 a\Lambda W_2^* W_3 + W_1(1 - W_1^2) - \frac{\nu}{\mu L^2}K_1^2 W_1, \tag{7}$$

$$\frac{dW_n}{dt'} = i\frac{u_0}{\mu L}K_n[a\Lambda W_{n+1}^* W_{n+2} + bW_{n-1}^* W_{n+1} - c\Lambda^{-1}W_{n-2}W_{n-1}] - \frac{\nu}{\mu L^2}K_n^2 W_n n > 1. \tag{8}$$

Equation (7) tells us that the dynamical behavior of u_n depends on two dimensionless numbers, namely, the Reynolds number $\mathrm{Re} \equiv u_0 L/\nu$ and the number $B \equiv u_0/(\mu L)$. We will investigate the statistical properties of Eq. (5) and (6) for $\mathrm{Re} \gg 1$ and for different values of B. In particular we fix $\mu = 1$ and u_0 real, while the parameters of the model are $a = 1$, $b = -0.4$, $c = 0.6$.

In Fig. 1 we show U_r as a function of time for $\nu = 10^{-6}$ and $B = 1$ ($u_0 = 2$), while in the insert of the same figure we plot U_r for $B = 2$. ($u_0 = 2$). As one can see, abrupt reversals of U_r are observed at apparently random times in both cases. The most important feature of Fig. 1 is that the characteristic correlation time of U_r is of order 1 (i.e., it is of order $L/u_0 \sim 1$), much smaller than the exit time $\langle\tau\rangle$. The behavior shown in Fig. 1 does not change by increasing the *Reynolds* number.

A more refined numerical simulation shows that the "random" exit times τ are distributed according to Eq. (3) with $\langle\tau\rangle \sim 600$ and $\langle\tau\rangle \sim 65$ for $B = 1$ and $B = 2$, respectively. In the insert of Fig. 2 we show $\log P(\tau)$ versus τ for the case $B = 2$, where the line shown in the figure represents the quantity $\exp(-\tau/65)$. We want to remark that the dynamics of U_r cannot be considered *slow* with respect to the characteristic time scale of u_2 and u_3. More precisely, let us consider the correlation functions of u_n. $C_n(T) \equiv \langle \mathrm{real}[u_n(t+T)]\mathrm{real}[u(t)]\rangle$. In Fig. 2 we show $C_n(T)$ for $n = 1, 2, 3$ for $B = 1$, computed

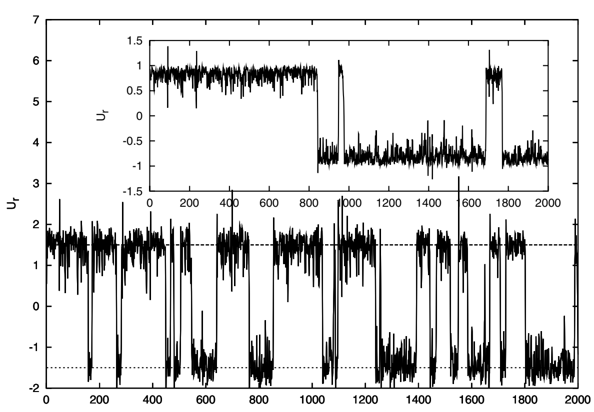

FIG. 1. The velocity U_r plotted as a function of time as obtained by numerical simulation of Eq. (5) for $u_0 = 2$, $\mu = 1$, and $\nu = 10^{-6}$. Inset: U_r as a function of time for $u_0 = 1$; the other parameters kept constant.

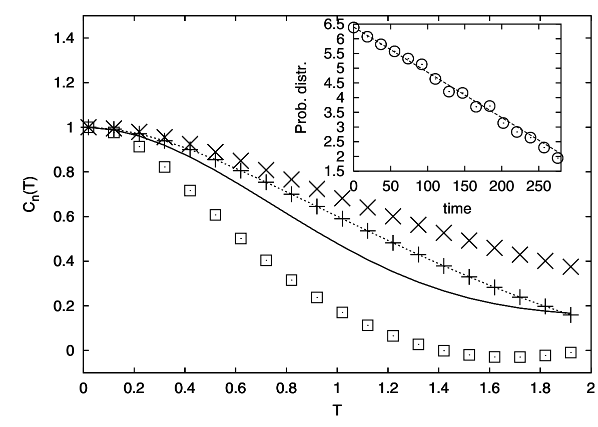

FIG. 2. The correlation functions $C_n(T) \equiv \langle \mathrm{real}[u_n(t + T)]\mathrm{real}[u_n(t)]\rangle$ plotted as a function of T for $n = 1$ (solid line), $n = 2$ (X), $n = 3$ (dotted line with +), and $B = 1$. The curve with squares shows the correlation function $C_\Phi(T)$. Inset: $\log[P(\tau)]$ as a function of τ as obtained by numerical simulations of (5) for $u_0 = 2$. According to the theory of stochastic differential equations, $P(\tau)$ is $\exp(-\tau/\langle\tau\rangle)$ where $\langle\tau\rangle$ is the average exit time. The solid line in the figure shows $\exp(-\tau/65)$, which is an extremely good fit to the observed $P(\tau)$.

by (ensemble) averaging when the system is near one of the two states; i.e., no reversal is included in the average. As one can clearly see, the characteristic time scales of u_2 and u_3 are similar to the characteristic time scale of u_1. Also, we can compute the correlation function of $C_\Phi(T) \equiv \langle \mathrm{real}[\Phi(t+T)]\mathrm{real}[\Phi(t)]\rangle$ of the nonlinear terms in Eq. (5). It is shown in Fig. 2 that the $C_\Phi(T)$ is decaying with a correlation time similar to those observed for $C_1(T)$ Thus, the nonlinear term in Eq. (5) cannot be considered a *fast* variable with respect to u_1. The same results hold for $B = 2$. For larger values of u_0, the average exit time $\langle\tau\rangle$ becomes smaller and eventually, for $u_0 \sim 10$ it becomes of order 1 (see also the discussion below). Our interest will focus on values of u_0 in the range [1,2] where $\langle\tau\rangle$ is at least 2 orders of magnitude longer than μ^{-1} and L/u_0, the two relevant deterministic time scales of Eq. (5). A closer look at Figs. 1 and 2 reveals that the two states of U_r are not $\pm u_0$. For $B = 1$, the maxima $\pm U_M$ in the probability distribution of U_r are located at $U_M = 0.84$, while for $B = 2$ we find $U_M = 1.5$, represented as horizontal lines in Fig. 2. In order to explain these results, let us consider more carefully the physical meaning of Φ in Eq. (5). The quantity $\mathrm{real}(\Phi u_1^*)$ is the amount of energy transferred by mode u_1 to smaller scales, i.e., u_2 and u_3. On the average, we know that $\mathrm{real}\langle(\Phi u_1^*)\rangle \equiv -\epsilon < 0$, where ϵ is the average rate of energy dissipation. Thus, as a first approximation, we can assume that

$$\Phi = -\beta u_1 + \phi' \tag{9}$$

where $\langle \phi' u_1^* \rangle = 0$ and $\beta > 0$. Equation (9) is consistent with the theory proposed in [11] and it should be considered in a statistical sense as the following discussion clarifies. In Fig. 3 we show the correlation functions $C_n(T)$, $C_\Phi(T)$ and $C_{\phi'}(T) \equiv \langle \mathrm{real}[\phi'(t+T)]\mathrm{real}[\phi'(t)]\rangle$, all of them computed by including in the (ensemble) average flow reversals. $C_2(T)$, $C_3(T)$, and $C_{\phi'}(T)$ decay quite rapidly with a correlation time of order 1. On the other hand, $C_1(T)$ *and* $C_\Phi(T)$ decay rather slowly with a correlation time of order $\langle\tau\rangle$, i.e., $\Phi(t)$ is correlated as $u_1(t)$ while $\phi'(t)$ is a *fast* variable with respect to both Φ and u_1. Note that there is no contradiction between the results shown in Figs. 2 and 3. When we observe the system around one of the two states and for a time interval not including flow reversals, all the large scale variables (u_1, u_2, u_3, and Φ) have almost the same correlation time. When we observe the system over a long time scale *including* flow reversals, the average exit time $\langle\tau\rangle$ determines the correlation time of u_1 and Φ and the other large scale variables can be considered to be *fast variables*. Figure 3 shows that Eq. (9) correctly tells us which are the slow components of Φ, namely $-\beta u_1$, and the *fast* components, i.e., ϕ'. Finally, let us remark that the time scale separations shown in Fig. 3 emerge spontaneously in the system as the result of the nonlinear energy flux from large to small scales.

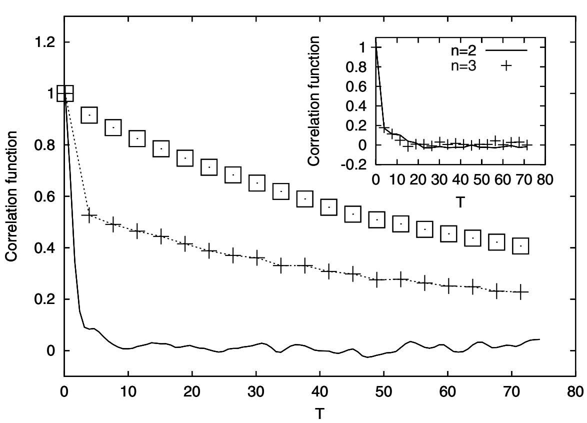

FIG. 3. The correlation functions $C_1(T)$ (squares) $C_\Phi(T)$ (lines with +), and $C_{\phi'}(T)$(solid line) as a function of T Inset: the correlation function of $C_2(T)$ and $C_3(T)$.

By multiplying both sides by u_1^* and taking the time average, we can compute β as

$$\beta = \mathrm{real}(\langle \Phi u_1^* \rangle)/\langle |u_1|^2 \rangle. \tag{10}$$

Thus, we must expect that two maxima in the probability distribution of U_r should corresponds to the solution of the equation

$$\left(\mu - \nu k_1^2 - \frac{\epsilon}{\langle |u_1|^2 \rangle}\right) U_r - \frac{\mu}{u_0^2} U_r^3 = 0. \tag{11}$$

Equation (11) tells us two important facts. First, the "states" $\pm U_M$, (between which abrupt transitions are observed) are not stationary solutions of the deterministic

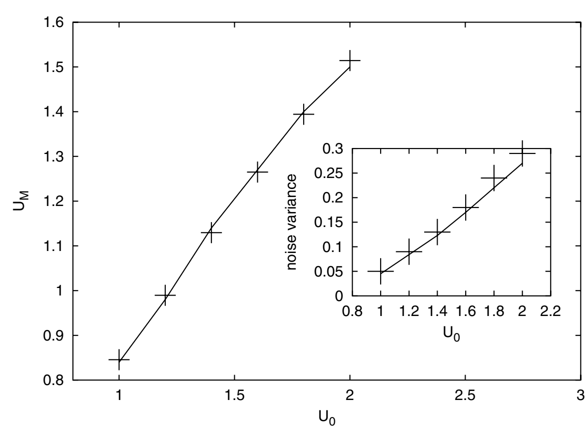

FIG. 4. Computing the statistically stationary solutions for U_r. The line corresponds to $u_0\sqrt{\mu - \beta(u_0) - \nu k_1^2}$, where $\beta(u_0)$ is defined by Eq. (10). The symbols represent the value of the maxima U_M of the probability density function of U_r obtained by the numerical simulations of (5) for different values of u_0. Insert: plot of $\sigma_{\delta U}$ (symbols) and σ_τ (solid line) for different values of u_0. σ_τ is the noise computed by Eq. (13) while $\sigma_{\delta U}$ is the noise computed from the fluctuations of U_r near the statistically stationary states, see Eq. (14).

Eq. (5), rather the states should be considered as *statistically stationary states* of the system. Second, we must expect $U_M < u_0$ as far as an energy cascade, $\epsilon > 0$, is produced. We have carefully checked the validity of Eq. (11) for u_0 in the range [1,2]. We have computed β from numerical simulations according to Eq. (10). It turns out that the numerical values of β are extremely well represented by the function $\beta(u_0) = 0.15 + 0.14u_0$. Next, we have solved Eq. (11) for each value of u_0 and $\beta(u_0)$ getting the line represented in Fig. 4. Finally we have computed the value of U_M using the probability distribution of U_r as obtained by the numerical simulations of the model. The results are shown as symbols in Fig. 4: the agreement is excellent. In light of the above results, it is tempting to argue that the behavior of (5) is consistent with the stochastic differential equation

$$du_1 = \left[(\mu - \beta(u_0))u_1 + \mu u_1 \frac{u_1^2}{u_0^2}\right]dt + \sqrt{\sigma}dW(t) \quad (12)$$

for a suitable value of the noise variance σ (hereafter we shall neglect the very small term νk_1^2). Note that we are claiming the validity of (12) over a time scale of order $\langle\tau\rangle$. Using (9) and (5), the quantity $\sqrt{\sigma}dW(t)$ represents the *deterministic* term $\phi'(t)dt$. In order to validate (12), we need to check whether the observed fluctuations of u_1 are in agreement with the average exit time $\langle\tau\rangle$. More specifically, using (12) and (2) we have

$$\langle\tau\rangle = \frac{\pi}{\sqrt{2}(\mu - \beta)} \exp[(\mu - \beta)^2 u_0^2/2\mu\sigma]. \quad (13)$$

By the numerical value of $\langle\tau\rangle$ and β, we can compute the intensity of the noise σ, hereafter denoted by σ_τ, needed to explain the average exit time obtained by the numerical simulations. The crucial point is whether the observed fluctuations of U_r near one the two states are ‘‘compatible’’ with the noise intensity σ_τ. To answer this question, let δU be small deviation around the statistically stationary states. Using (12) we can compute $\langle(\delta U)^2\rangle$ as

$$\langle(\delta U)^2\rangle = \frac{\sigma}{4(\mu - \beta)}. \quad (14)$$

By using the numerical simulations, we can estimate $\langle(\delta U)^2\rangle$ around the statistically stationary states. Finally, using (14), we can estimate the noise variance σ, hereafter referred to as $\sigma_{\delta U}$, which explains the observed fluctuations of δU. For Eq. (12) to represent a good approximation of the full nonlinear deterministic system, we must obtain

$$\sigma_{\delta U} \sim \sigma_\tau. \quad (15)$$

In the inset of Fig. 4, we plot σ_τ (line) and $\sigma_{\delta U}$ (crosses) for different values of u_0 in the range [1,2]. As one can see, (15) is verified with very good accuracy, with at most *10 percent* difference for $u_0 = 2$. We remark that the result shown in Fig. 4 represents a rather severe test on the validity of Eq. (12).

We want finally to comment on the behavior of the model in the limit of large u_0 or, equivalently, in the limit of large B. For large value of u_0 we reach the condition $\beta(u_0) \sim \mu$. Actually, arguments based on energy balance and numerical simulations clearly show that the maximum value of β is 1. In this case, Eq. (11) gives $U_M \sim 0$, i.e., the two ‘‘statistically stationary states’’ disappear. Also, the quantity $\langle\tau\rangle[\mu - \beta(u_0)]$ becomes order 1 for large u_0 and the there is no time scale separation between the average exit times and the eddy turnover time of the system. It follows that, for large B, one cannot speak of ‘‘abrupt flow reversal.’’

To our knowledge, the results shown in this Letter clearly demonstrate, for the first time, that Eq. (1) can be regarded as a very good approximation for the long time (large deviation) dynamics of a deterministic dynamical system. We argue that some of the experimental results mentioned in the introduction can be investigated and explained in the framework of stochastic differential equations even if no time scale separation exists between the ‘‘wind’’ fluctuations and turbulence.

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