A Class of Time-Machine Solutions with a Compact Vacuum Core

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We present a class of curved-spacetime vacuum solutions which develop closed timelike curves at some particular moment. We then use these vacuum solutions to construct a time-machine model. The causality violation occurs inside an empty torus, which constitutes the time-machine core. The matter field surrounding this empty torus satisfies the weak, dominant, and strong energy conditions. The model is regular, asymptotically flat, and topologically trivial. Stability remains the main open question.

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The problem of time-machine formation is one of the outstanding open questions in spacetime physics. Time machines are spacetime configurations including closed timelike curves (CTCs), allowing physical observers to return to their own past. In the presence of a time machine, our usual notion of causality does not hold. The main question is: Do the laws of nature allow, in principle, the creation of a time machine from ''normal'' initial conditions? (By normal I mean, in particular, initial state with no CTCs.)

Several types of time-machine models were explored so far. The early proposals include Godel's rotating-dust cosmological model [1] and Tipler's rotating-string solution [2]. More modern proposals include the wormhole model by Moris, Thorne, and Yurtsever [3], and Gott's solution [4] of two infinitely long cosmic strings. Ori [5] later presented a time-machine model which is asymptotically flat and topologically trivial. Later Alcubierre introduced the warp-drive concept [6], which may also lead to CTCs.

The above models, however, all suffer from one or more severe problems. In some of them [3,6] the weak energy condition (WEC) [7] is violated, indicating unrealistic matter-energy content. The WEC states that for any physical (timelike) observer the energy density is non-negative, which is the case for all known types of (classical) matter fields. In other models the CTCs are either preexisting [1,2,4] and/or ''come from infinity'' [4] (see [8,9]), and/or there is a curvature singularity [2].

The only of the above models which does not violate the WEC, and in which CTCs evolve from normal initial data (and within a compact region of space) is that of Ref. [5]. But this model, too, is not satisfactory, for the following reason: The energy-momentum source occupying the time-machine core, though consistent with the WEC (and also with the dominant energy condition [10]), does not fit any known type of matter field. Therefore this configuration (spacetime plus matter) is not a solution of any prescribed set of field equations. Even if we assume there exist physical matter fields that yield the desired initial configuration, we cannot tell how these fields (and the geometry) will evolve in time, because we do not have at hand the set of evolution equations. It therefore leaves open the question of whether the system will or will not form CTCs. All we can say is that the WEC does not preclude the formation of CTCs.

We would therefore like a good time-machine model to be made of a well-known matter field, preferably an elementary one. Obviously the most elementary field (in the present context) is the pure gravitational field, i.e., a vacuum spacetime. It is primarily this issue which we address in the present paper.

There are several other desired features which we would like our model to satisfy: We want the spacetime (and particularly the initial hypersurface) to be asymptotically flat (or, alternatively, asymptotically Friedmann). In addition, we would like the onset of causality violation to take place within a *finite* region of space. That is, we would like the initial hypersurface to include a compact piece S_0, such that the onset of causality violation—the appearance of a closed causal loop—will be fully dictated by the initial conditions on S_0. This requirement makes sense, because presumably even an advanced civilization will not be able to control the initial data in the entire space, but only in a finite region (in the best case). Note that this criterion for compactness differs from the notion of ''compact generation'' introduced by Hawking [11], as we further discuss below. We shall refer to the compact region of spacetime including S_0 and also its future domain of dependence and its neighborhood, as the *core* of the time-machine spacetime (in our model this will include the first causal loop and the CTCs in its immediate neighborhood).

In this Letter we present a class of vacuum solutions in which CTCs form at some particular moment. We then use such a vacuum solution as the core of a new type of time-machine model which satisfies all the above requirements. In this model the evolution starts from a regular initial spacelike hypersurface (partial Cauchy surface) which (like spacetime itself) is asymptotically flat, topologically trivial, and satisfies the weak, dominant, and strong energy conditions (the *energy conditions*). Then CTCs form in the central region at some particular moment.

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Our model consists of an empty (i.e., vacuum) torus which constitutes the time-machine core. This toroidal vacuum region is immersed in a larger, spherelike, region of matter satisfying the energy conditions. The matter region is finite, and is surrounded by an external asymptotically flat vacuum region (specifically in our construction it is the Schwarzschild geometry). The matter in the intermediate range, though satisfying the energy conditions, is not associated as yet with any specific known matter field (though it appears likely that it will be possible to compose it from, e.g., some combination of neutral and charged dusts, and electromagnetic fields). Hence we cannot tell yet what is the matter's evolution equation. Nevertheless, the causality violation occurs in the internal vacuum core, and it is completely dictated by the initial conditions at the compact set S_0 (located in the vacuum core too). Here we shall primarily discuss the vacuum solution at the time-machine core (which is the main new feature in this Letter); the structure of the surrounding matter and its matching to the external asymptotically flat universe will be presented elsewhere [12].

There are two rather general analyses, by Tipler [13] and by Hawking [11], which put constraints on the creation of a time machine in a compact region of space without violating the WEC. At first sight each of these analyses might appear to preclude a model like the one presented here. A closer look, however, reveals that there is no inconsistency. This was already demonstrated and explained in Ref. [5] (which, too, presents a time-machine model satisfying the WEC). In short, Hawking's analysis refers to *compactly generated* time machines, which probably is not the case here (see below). Our model is consistent with Tipler's analysis because it includes a closed null geodesic (denoted N below) which is future incomplete (though no local irregularity occurs there; See also the discussion in Ref. [5]).

We turn now to describe the geometry of the time-machine core. Consider the vacuum solution

$$ds^2 = dx^2 + dy^2 - 2dzdT + [f(x, y, z) - T]dz^2. \quad (1)$$

The coordinates (x, y, T) get all real values (though we later truncate the solution in x and y at the internal boundary of the matter region), but z is a periodic coordinate, $0 \le z \le L$ for some $L > 0$, with $z = L$ and $z = 0$ identified. f is any function (properly periodic in z) satisfying

$$f_{,xx} + f_{,yy} = 0. \quad (2)$$

[For f not satisfying Eq. (2) the metric (1) represents null dust with G_{zz} given by $-(1/2)(f_{,xx} + f_{,yy})$ and all other components vanishing. We shall not discuss this case here.] [14] It can be shown that this class is locally isometric to vacuum plane-fronted waves [15]. However, its global properties are completely different from those of plane-fronted waves, as we now discuss.

One immediately observes that the metric (1) develops CTCs at sufficiently large T. For each x, y, the metric function $g_{zz} = f(x, y, z) - T$ is positive (for all z) at small T, but becomes negative (for all z) at sufficiently large T. Consequently the closed curves of constant x, y, T are spacelike at small T but become timelike—namely CTCs—at large T. (Below we consider a specific example in more detail.) Note that the metric is everywhere regular with $\det(g) = -1$, so no local pathology is involved in the formation of CTCs.

For generic f the spacetime is curved, with

$$R_{izjz} = -(1/2)f_{,ij} \quad (3)$$

(and its obvious permutations), and all other components vanishing, where i, j stand for x and y. The metric becomes locally flat in the degenerate case $f = 0$ [the same holds for any $f = f(z)$]. This is just the Misner space [16], generalized to four dimensions in a straightforward manner. The degenerate $f = 0$ spacetime shares some of the features of the generic-f nondegenerate case, though its global structure is different and more pathological (see below).

For concreteness we now specialize to a simple example. We take

$$f = a(x^2 - y^2)/2 \quad (4)$$

for some constant $a > 0$. This yields an empty curved spacetime, locally isometric to a linearly polarized plane wave. We now transform from T to a new time coordinate

$$t = T - a(x^2 - y^2)/2 + e\rho^2$$

for some $e > 0$, where $\rho^2 \equiv x^2 + y^2$. This turns out to be convenient, because the hypersurfaces $t = \text{const}$ provide a useful foliation for the analysis below. The line element becomes

$$ds^2 = dx^2 + dy^2 - 2dzdt + (e\rho^2 - t)dz^2 + 2[(2e - a)xdx + (2e + a)ydy]dz. \quad (5)$$

This metric is empty and curved (as before), and again one finds $\det(g) = -1$. Since $g_{zz} = e\rho^2 - t$, the closed curves of constant x, y, t are spacelike at $t < e\rho^2$ and timelike at $t > e\rho^2$ (and null at $t = e\rho^2$). These curves are nongeodesic, except for the single closed null geodesic $x = y = t = 0$, which we denote N.

A hypersurface $t = \text{const}$ is spacelike when $g^{tt} < 0$ and timelike whenever $g^{tt} > 0$. One finds

$$g^{tt} = t + (2e - a)^2x^2 + (2e + a)^2y^2 - e\rho^2.$$

Choose now sufficiently small positive a and e such that $e > (2e + a)^2$. Then the hypersurfaces $t = \text{const} \le 0$ may be characterized as follows: (I) For $t < 0$ they are spacelike throughout; (II) The hypersurface $t = 0$ is spacelike everywhere, except at the central curve, $x = y = 0$ (the geodesic N), where it is null ($g^{tt} = 0$).

(For $t > 0$ the hypersurfaces $t = \text{const}$ are mixed: timelike for sufficiently small ρ and spacelike at large ρ.)

In the degenerate case $f = 0$ (the 4d Misner space), the hypersurface $T = 0$ is null, and all its generators are closed null geodesics (the curves of constant x, y). This case is pathological, because the analytic extension beyond $T = 0$ is nonunique [16]. No such pathology occurs in the nondegenerate case. For a generic f, the closed null geodesics are isolated [like N in the specific example (4)] and do not form a hypersurface. Note also that in the degenerate case $f = 0$ the causal structure is inherently unstable in the following sense: Adding an arbitrarily small fixed number to g_{xz} and/or g_{yz} (which vanish otherwise) leaves the geometry vacuum and locally flat, but the hypersurface of closed null geodesics entirely disappears. [This metric is isometric to (part of) the Grant spacetime [9], which includes no closed causal geodesics.] Such a global instability does not occur in the nondegenerate case, which we consider throughout this Letter.

The above constructed vacuum solution is now used as the core of our time-machine spacetime, with $x = y = 0$ located at the central circle of the torus. The solution (5) is truncated at $\rho = \rho_0$ for some $\rho_0 > 0$, where the matter region starts.

As was discussed above, we would like the region of CTCs to evolve, in a deterministic manner, from a compact spacelike hypersurface S_0 in the vacuum core. This would be fully accomplished if a region of CTCs were included in $D^+(S_0)$ (D^+ denotes the future domain of dependence [7], also known as the future Cauchy development). But obviously this can never be the case because by definition D^+ of any spacelike hypersurface cannot include any closed causal curve. It may be possible, however, that the *boundary* of $D^+(S_0)$ will include a closed causal curve (in fact a closed null geodesic); And this appears to be the maximum one can hope for, in terms of the causal relation between S_0 and the region of causality violation. We shall now show that our model indeed has this desired feature, namely, one can choose a compact spacelike hypersurface S_0 in the vacuum core, such that a closed causal curve (the null geodesic N) appears at the boundary of $D^+(S_0)$. Furthermore, any regular extension of the geometry beyond N will include a region of CTCs.

For any $\rho_1 > 0$ the hypersurfaces $t = \text{const} \leq 0$ are all spacelike at $\rho \geq \rho_1$. Also, the hypersurfaces $t = \text{const} < 0$ are spacelike even at $\rho < \rho_1$. Any such spacelike hypersurface may be slightly deformed and still remain spacelike. Consider, in particular, the composed hypersurface given by $t = \text{const} = t_0 < 0$ at $\rho \leq \rho_1$, by $t = 0$ at $\rho \geq \rho_2$, and by some interpolating function $t = \tilde{t}(\rho)$ in the range $\rho_1 \leq \rho \leq \rho_2$, for some parameters $0 < \rho_1 < \rho_2 < \rho_0$. The function $\tilde{t}(\rho)$ is chosen to be a monotonic one, which smoothly bridges between $t = t_0$ at $\rho = \rho_1$ and $t = 0$ at $\rho = \rho_2$. For sufficiently small $|t_0|$, the function $\tilde{t}(\rho)$ may be taken to be of sufficiently small slope, such that the hypersurface described by $t = \tilde{t}(\rho)$ is spacelike in $\rho_1 \leq \rho \leq \rho_2$. By this we have constructed a composed spacelike hypersurface in the entire range $\rho \leq \rho_0$. We then restrict this hypersurface to the range $\rho \leq \rho_3$ for some $\rho_2 < \rho_3 < \rho_0$, and denote this compact spacelike hypersurface by S_0.

Take now any point P at $t < 0$ in a sufficiently small neighborhood of N (located at $\rho = t = 0$), and consider any inextendible past-directed causal curve γ emanating from P. (In particular P is located at $\rho < \rho_1$ and $t > t_0$, i.e., at the future of S_0.) Since in the range $t_0 \leq t < 0$ all hypersurfaces $t = \text{const}$ are entirely spacelike, one can easily show that γ has no other choice but to move towards smaller t values, until it intersects S_0. Therefore the point P belongs to $D^+(S_0)$. This means that N resides at the boundary of $D^+(S_0)$.

The compact initial hypersurface S_0 can be extended through the matter region and the external vacuum region to spacelike infinity, to form a regular, asymptotically flat, partial Cauchy surface, everywhere satisfying the energy conditions, which we denote Σ. This will be demonstrated elsewhere [12].

We conclude that in our model the onset of causality violation—the appearance of the closed null geodesic N—is fully dictated by the initial conditions in the compact vacuum piece S_0 of the initial hypersurface Σ. In particular, the details of the initial data at the external vacuum region, or at the intermediate matter region, cannot interfere with this onset of causality violation. Although the above construction only demonstrates the inevitable formation of a single closed causal orbit, any smooth (C^1 metric) extension of the geometry at N to $t > 0$ will include a region of CTCs, because $g_{zz,t} < 0$ at N.

Hawking [11] previously introduced another notion of compactness: A *compactly generated* time machine is one in which all null generators of the Cauchy horizon, when traced back to the past, enter a compact region of spacetime and remain there. He then showed that a compactly generated time machine (with noncompact initial hypersurface) must include a region where the WEC is violated. Our model evades this WEC violation, probably because it is not compactly generated. In the intermediate matter region, the initial data for the metric are constructed to be regular and smooth (and to satisfy the WEC) [12]. This holds on Σ and also for some time interval to its future. But our construction by no means guarantees that the regular WEC-satisfying metric at the matter region can extend to all future times. From Hawking's arguments [11] it appears to follow that the matter geometry, when forced to satisfy the WEC, must develope some noncompactness (e.g., a singularity or an ''internal infinity'') at later times. Then these arguments, applied to our construction, may further suggest that generators of the chronology horizon will emanate from this noncompact region. But this proposed scenario must be verified by an explicit study of the structure of the Cauchy horizon. We emphasize again that although probably not "compactly generated," our model does demonstrate the formation of closed causal loops from the initial data on a compact vacuum region S_0. Although it is only a single closed null geodesic which resides in the closure of $D^+(S_0)$, any smooth (C^1) extension of the metric to $t > 0$ will include a region of CTCs.

The fact that the weak, dominant, and strong energy conditions are satisfied suggests that the initial conditions required for our model are physically acceptable. This does not mean that we shall be able to practically initiate such initial conditions in the foreseeable future. However, perhaps an advanced civilization will be able to do this (and perhaps even natural processes, involving large gravitating masses in rapid motion, may lead to such conditions). There still remains, however, the issue of *stability*. Several analyses [11,17,18] indicated possible instabilities of various time-machine solutions to classical perturbations and/or quantum-mechanical fluctuations (see however [19]). Whereas these analyses mostly referred to compactly generated models, some of the arguments for quantum instabilities apply to noncompactly generated models as well (see, in particular, the discussion in [18]). These instabilities may raise doubts on whether a model like the one presented here can be implemented in reality. Yet, it appears that so far these indications for instability do not rule out the possibility of actual time-machine construction. The strength of the quantum instability is not clear yet, and, more importantly, it is not known yet what will be the *outcome* of this instability (namely, what will be the spacetime configuration that eventually forms). Perhaps we shall have to await the formulation of the full theory of quantum gravity before we know whether quantum instabilities provide chronology protection or not (see discussion in [20]). The situation with regards to classical instabilities is somewhat different: Their occurrence still needs be established, especially in the noncompactly generated case and beyond the context of geometrical optics. If classical instabilities are found to be inevitable, their strength and their outcome can be explored by evolving the classical Einstein equations for slightly perturbed initial conditions.

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