Stability of Fermi Surfaces and K Theory

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Nonrelativistic Fermi liquids in $d + 1$ dimensions exhibit generalized Fermi surfaces: $(d - p)$-dimensional submanifolds in the $(\mathbf{k}, \omega)$-space supporting gapless excitations. We show that the universality classes of stable Fermi surfaces are classified by K theory, with the pattern of stability determined by Bott periodicity. The Atiyah-Bott-Shapiro construction implies that the low-energy modes near a Fermi surface exhibit relativistic invariance in the transverse $p + 1$ dimensions. This suggests an intriguing parallel between nonrelativistic Fermi liquids and D-branes of string theory.

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The main focus of this Letter is nonrelativistic condensed matter systems. However, its primary motivation comes from recent findings in superstring theory, in particular, the physics of D-branes. (Readers not interested in the string theory motivation can safely skip the next six paragraphs.)

D-branes are nonperturbative, inherently stringy solitonic defects in Type I and Type II superstring theory, defined as hypersurfaces in space-time on which strings can end [1]. They naturally couple [2] to higher-form gauge fields (Ramond-Ramond fields) in superstring theory, in a manner analogous to how charged particles couple to the electromagnetic potential.

Given a space-time background Y that solves string theory, it is natural to attempt a classification of D-brane charges on Y. The Ramond-Ramond (RR) fields that D-branes couple to have traditionally been viewed as differential forms on Y. At low energies, the theory is described by a supergravity theory, whose action is schematically given by [3]

$$\mathcal{L} \sim \int_Y d^{10}x\sqrt{g}\left(R - \sum_p \frac{1}{p!} dC_p \wedge \star dC_p + \cdots\right). \quad (1)$$

The allowed degrees p of the RR forms C_p depend on the type of the superstring theory considered. Thus, it may seem natural to expect that charges of D-branes take values in the de Rham cohomology $H^*(Y, \mathbf{R})$, or perhaps its generalization over $\mathbf{Z}$ when the Dirac charge quantization is taken into account.

This expectation is now known to be incorrect [4–6]. D-branes are stringy objects, and not just submanifolds in space-time, and open string excitations give the D-brane world-volume Σ extra structure. In particular, $\mathcal{N}$ coincident D-branes wrapping Σ are characterized by a $U(\mathcal{N})$ gauge bundle on Σ. When these facts are properly taken into account, it turns out that D-brane charges on a given manifold Y take values in K-theory groups of Y. K theory (see, e.g., [7,8] for some background) is an exotic cohomology theory, built from equivalence classes of $U(\mathcal{N})$ bundles on Y, and closely related to but distinct from the conventional cohomology $H^*(Y, \mathbf{Z})$.

This implies that the RR fields are also objects in K theory, and not differential forms [9], making the low-energy description of string theory on Y in terms of (1) questionable. The operations needed in (1)—the exterior derivative d, the wedge product $\wedge$, and the Hodge star $\star$—are well defined on differential forms, but once the C_p's are reinterpreted as K-theory objects, it is not clear how to even define (1). This crisis of the Lagrangian formulation of low-energy string theory is further supported by the discovery [10] of apparently non-Lagrangian phases in the partition functions of various string and M-theory vacua.

Perhaps this means that the Lagrangian framework currently available is insufficient for RR fields, but its suitable generalization awaits to be discovered. (Important steps in this direction have been taken [11].) Alternatively, the theory may require a non-Lagrangian formulation. (This may already be suggested by the presence of a self-dual RR field strength in Type IIB theory). Before we settle on either of these two alternatives, however, we should consider a third possibility. The subtle K-theory features of string theory could be an emergent phenomenon, with D-branes and RR fields emerging as composites of some more elementary degrees of freedom that admit a conventional Lagrangian description.

One purpose of this Letter is to demonstrate that in another area of physics—namely, in nonrelativistic condensed matter—a close analog of this third possibility is realized: Objects classified by K theory do emerge as derived objects in systems whose microscopic degrees of freedom conform to a very conventional Lagrangian path-integral description. In the process, we learn potentially valuable lessons for condensed matter systems, and discover an intriguing parallel between Fermi liquids and D-branes of string theory.

In the rest of this Letter, we study nonrelativistic Fermi liquids in $d + 1$ space-time dimensions. Usually, a restriction to $d \leq 3$ quickly follows, but we keep d arbitrary. This makes some of the patterns more transparent; moreover, string theory is itself naturally formulated in space-times of higher dimensionality.

We are interested in the low-energy dynamics of long-wavelength excitations in a Fermi liquid, described microscopically by a complex fermion $\psi^{i\sigma}(\mathbf{x}, t)$. σ is the spinor index of the $SO(d)$ rotation group, and i is an internal index of an n-dimensional representation of some compact symmetry G. For simplicity, we consider i the fundamental of $SU(n)$, the generalization to any compact G being straightforward. Since the irreducible spinor representation of $SO(d)$ is $2^{[d/2]}$ dimensional (with $[k]$ denoting the integral part of k), ψ has $N \equiv 2^{[d/2]}n$ complex components. Real fermions are considered in the closing part of this Letter.

The theory is microscopically described by the following nonrelativistic Lagrangian,

$$S = -\int dt d^d\mathbf{x}\left(i\psi^\dagger_{i\sigma}\partial_t\psi^{i\sigma} + \frac{1}{2m}\psi^\dagger_{i\sigma}\Delta\psi^{i\sigma} - \mu\psi^\dagger_{i\sigma}\psi^{i\sigma}\right) + \cdots.$$

Here Δ is the Laplace operator in d dimensions, μ is the chemical potential, and "..." denotes all interactions, such as those with an order parameter, four-Fermi interactions, averaged impurities, etc.

We wish to identify possible universality classes of the renormalization group (RG) behavior and the corresponding RG fixed points. The essence of the Landau theory of Fermi liquids [12,13] is in the observation that the low-energy excitations of the system populate a narrow shell near the Fermi surface, suggesting a coarse-graining procedure whereby momenta are scaled towards the Fermi surface. Free fermions represent the simplest fixed point of this scaling. The Fermi surface supports gapless coarse-grained fermionic excitations χ, with the characteristic linear dispersion relation $\omega \sim k + \cdots$ at low energies.

Once a free-field fixed point has been identified, a full RG analysis reveals possible instabilities of the system, due to (marginally) relevant interactions. The degeneracy of the gapless modes χ can be lifted completely, and the system can develop a gap (such as in the case of superconductivity, or in the B phase of ^{3}He). Alternatively, the gapless excitations can be only partially lifted, leaving a submanifold of some lower dimension $(d - p)$ in the $(\mathbf{k}, \omega)$ space where gapless fermionic excitations are supported. Since such submanifolds with gapless excitations generalize the concept of the Fermi surface, we refer to them as "generalized Fermi surfaces" (and drop the word "generalized" from now on).

Perhaps the standard example of a partially lifted Fermi surface is the A phase of ^{3}He in $d = 3$, where the interactions with the order parameter cause the Fermi modes to become massive outside of isolated points. Such Fermi points are stable under small perturbations of the system. Thus, in $d = 3$, we can have stable Fermi surfaces (as in the free fermion system) or Fermi points (as in ^{3}He-A); however, Fermi *lines* appear to be (generically) unstable (see, e.g., [14]). We now argue that this is the beginning of a pattern, explained as a consequence of Bott periodicity in K theory (see [7,8] for reviews on K theory).

To see that K theory controls the stability of Fermi surfaces, we study the exact propagator in Euclidean time (and in the obvious notation),

$$G^{i\sigma}_{i'\sigma'}(\mathbf{k}, \omega) = \langle\psi^{i\sigma}(0,0)\psi^\dagger_{i'\sigma'}(\mathbf{k}, \omega)\rangle. \tag{2}$$

Gapless excitations correspond to a pole in G along some submanifold Σ, of some dimension $d - p$, in the $(\mathbf{k}, \omega)$ space. We analyze the stability of such poles under small perturbations in the theory, i.e., perturbations that do not change G qualitatively far from Σ. (The analysis of large perturbations, corresponding to possible instabilities due to relevant deformations of the fixed point, is outside the scope of this Letter.)

It is convenient to introduce a collective index $(i\sigma) \equiv a$, $a = 1, \ldots, N$ with $N = 2^{[d/2]}n$, and consider the inverse exact propagator

$$\begin{aligned}\mathcal{G}_a{}^{a'} &\equiv (G^{-1})_a{}^{a'}(\mathbf{k}, \omega)\\ &= \delta^{a'}_a(i\omega - \mathbf{k}^2/2m + \mu) + \Pi_a{}^{a'}(\mathbf{k}, \omega),\end{aligned} \tag{3}$$

with $\Pi_a{}^{a'}(\mathbf{k}, \omega)$ the exact self-energy tensor. The question of stability of the manifold Σ of gapless modes reduces to the classification of the zeros of the matrix $\mathcal{G}$ that cannot be lifted by small perturbations of $\Pi_a{}^{a'}$. Our arguments are topological, and our results thus independent of the precise details of $\Pi_a{}^{a'}$.

Assume that $\mathcal{G}$ has a zero (i.e., det$\mathcal{G}$ vanishes) along a submanifold Σ of dimension $d - p$ in the $(d + 1)$-dimensional $(\mathbf{k}, \omega)$ space. Σ lies within the subspace of zero frequency, $\omega = 0$. Pick a point $\mathbf{k}_F$ on the Fermi surface Σ, and consider the $p + 1$ dimensions $\mathbf{k}_\perp$ transverse to Σ in the $(\mathbf{k}, \omega)$ space at $\mathbf{k}_F$. A small perturbation of the system can either move the zero of $\mathcal{G}$ slightly away from $\mathbf{k}_F$ along $\mathbf{k}_\perp$ or eliminate the zero altogether. In the latter case, the purported Fermi surface is unstable, and a small perturbation either produces a gap or at least further reduces the dimension of the Fermi surface. In order to classify *stable* zeros, consider a sphere S^p wrapped around Σ at some small distance ϵ in the transverse $p + 1$ dimensions $\mathbf{k}_\perp$, $|\mathbf{k}_\perp - \mathbf{k}_F| = \epsilon$. We assume ϵ small enough so that this S^p does not intersect any other components of the Fermi surface, a situation that can always be arranged in a generic point on the Fermi surface. The matrix $\mathcal{G}$ is nondegenerate along this S^p, and therefore defines a map

$$\mathcal{G} : S^p \to GL(N, \mathbf{C}) \tag{4}$$

from S^p to the group of nondegenerate complex $N \times N$ matrices. If this map represents a nontrivial class in the pth homotopy group $\pi_p(GL(N, \mathbf{C}))$, the zero along Σ cannot be lifted by a small deformation of the theory. The Fermi surface is then stable under small perturbations, and the corresponding nontrivial element of $\pi_p(GL(N, \mathbf{C}))$ represents the topological invariant (or "winding number") responsible for the stability of the Fermi surface.

One may worry that this could result in a very complicated pattern, dependent on the specific values of p and N. Fortunately, this is not the case, and the pattern that emerges is quite simple. The key observation is that the values of N and p are always in the so-called stable regime [7], of N large enough so that $\pi_p(GL(N, \mathbf{C}))$ is independent of N. This stable regime lies at $N > p/2$ [7]. It is easy to check that in our setting, the stability condition is always satisfied.

It is a deep mathematical result that in this stable regime, the homotopy groups of $GL(N, \mathbf{C})$ define a generalized cohomology theory, known as K theory [7]. In K theory, any smooth manifold X is assigned an Abelian group $K(X)$. [For X noncompact, $K(X)$ is to be interpreted as compact K theory [7].] For example, for $X = \mathbf{R}^k$ this group is given by

$$K(\mathbf{R}^k) = \pi_{k-1}(GL(N, \mathbf{C})) \tag{5}$$

with N in the stable regime. The corresponding groups are known to be $K(\mathbf{R}^{2\ell}) = \mathbf{Z}$ and $K(\mathbf{R}^{2\ell+1}) = 0$. This periodicity by two is known as Bott periodicity [7].

Our analysis of the Fermi surface can now be reinterpreted as a statement about K theory: The map (4) defines an element of the K theory group $K(\mathbf{R}^k)$ of the transverse space $\mathbf{k}_\perp$; the Fermi surface Σ is stable if this element is nontrivial. We have established our first result.

Stable Fermi surfaces in Fermi liquids are classified by K theory; in the case of complex fermions, Fermi surfaces of codimension $p + 1$ in the $(\mathbf{k}, \omega)$ space are stable for p odd, and unstable for p even. In $d = 3$, this reproduces the observed pattern of stability mentioned above.

As our first application of this picture, we use K theory to determine the dispersion relation of the gapless modes near a general stable Fermi surface Σ. Such modes are described by coarse-grained fermions $\chi^\alpha(\omega, \mathbf{p}, \theta)$, with θ denoting coordinates on Σ, and $\mathbf{p}$ being the spatial momenta normal to Σ. The index α goes over some subset, to be determined below, of the range of a. The leading quadratic part of the action is

$$S = \int d\mu(\omega, \mathbf{p}, \theta)[\chi_\alpha^\dagger \mathcal{D}^\alpha_\beta(\omega, \mathbf{p}, \theta)\chi^\beta + \cdots]. \tag{6}$$

$d\mu(\omega, \mathbf{p}, \theta)$ is the flat measure $d\omega d^d\mathbf{k}$ written in terms of the new coordinates $(\omega, \mathbf{p}, \theta)$, and "..." refers to interactions of χ^α, to be studied by RG methods in the vicinity of the free-field fixed point given by (6). $\mathcal{D}$ is the operator we now wish to determine.

K theory provides an explicit construction of the generator e in $\mathrm{K}(\mathbf{R}^{2\ell})$, known as the Atiyah-Bott-Shapiro (ABS) construction [7,8,15]. Consider a stable Fermi surface Σ, of codimension $p + 1 \equiv 2\ell$, with winding number one. Any Σ with a higher winding number n can be perturbed into n separated Fermi surfaces of winding number one. The range N of the index a carried by the microscopic fermion is in the stable regime (and therefore quite large), but the range $\tilde{N}$ of the index α carried by the coarse-grained fermions χ^α can, in principle, be lower than N. The ABS construction determines the universal value of $\tilde{N}$ to be $\tilde{N} = 2^{[p/2]}$. If p_μ ($\mu = 0, \ldots, p$) are the dimensions transverse to Σ in $(\mathbf{k}, \omega)$, we first define the gamma matrices Γ^μ of $SO(p, 1)$ to satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta_{\mu\nu}$ a (Lorentz-signature) metric. The ABS construction then gives the leading expression for the topological invariant e, and hence for the inverse propagator $\mathcal{D}$ of the coarse-grained fermions χ^α near the Fermi surface,

$$\mathcal{D} = \Gamma^\mu p_\mu + \cdots. \tag{7}$$

The "$\cdots$" in (7) refers to higher-order corrections to the leading low-energy term. The precise form of the metric $\eta_{\mu\nu}$ is determined by the microphysics of ψ. This establishes our second result.

At low energies, the dispersion relation of the coarse-grained gapless fermions χ^α near the Fermi surface is governed by the Atiyah-Bott-Shapiro construction of K theory.

The ABS construction has determined the universal value of the range $\tilde{N}$ of α, making χ^α automatically a spinor of the Lorentz group $SO(p, 1)$. (Notice that $\tilde{N} \equiv 2^{[p/2]} \leq N$, with the equality only when $n = 1$ and $p = d$.) The equation of motion of χ^α *in the low-energy regime* is the relativistic Dirac equation, $\Gamma^\mu \partial_\mu \chi = 0 + \cdots$, with "$\cdots$" now denoting possible nonrelativistic corrections at higher energies. In the free theory, we get one copy of the Dirac fermion for each point (or patch) θ on the Fermi surface. This is our third result.

The low-energy modes exhibit an emergent relativistic dispersion relation, in the $p + 1$ dimensions transverse to the Fermi surface.

Again, some low-dimensional examples of this behavior are well known. Here we have extended the statement to arbitrary dimensions, and found the emergent relativistic invariance as a simple consequence of the ABS construction. Notice that both the gamma matrices *and* the fact that the coarse-grained fermions χ^α transform as spinors of $SO(p, 1)$ are emergent properties. In particular, the spin-statistics theorem familiar from relativistic quantum field theory emerges naturally.

The ABS construction played a crucial role in string theory, in Sen's picture [16] of stable D-branes as topological defects in the tachyon on higher-dimensional D-branes. In the string theory literature, this construction is sometimes referred to as the "Γx construction," since x are the space-time dimensions transverse to the soliton inside the higher-dimensional branes. In the theory of Fermi liquids, we have put the Γx construction where it naturally belongs: in the momentum space.

So far we have analyzed the theory locally near a point $\mathbf{k}_F$ on the Fermi surface. K theory provides a natural extension of our arguments globally, to a Fermi surface of arbitrary topology. Our construction demonstrates that a Fermi surface Σ is stable precisely when it carries a topological charge in K theory. Fermi surfaces are objects in K theory and not just submanifolds in the $(\mathbf{k}, \omega)$ space. K theory thus provides the natural arena for understanding the structure of Fermi surfaces. We expect it to be particularly useful in the case of Fermi surfaces with complicated topologies and/or singularities. It could also be a natural tool for understanding the ideas of topological order in Fermi systems [17].

The case of real fermions is perhaps even more interesting. Repeating the above steps for $\psi^{i\sigma}$ satisfying a reality condition $\psi^* \sim \psi$, we are naturally led to the classification of stable Fermi surfaces in terms of real KR theory [7]. The simplest reality condition in K theory defines what is known as KO theory, related to the homotopy groups of $GL(N, \mathbf{R})$. The subtlety here is that the involution that defines the reality condition on ψ acts simultaneously as the time reversal symmetry, so as to preserve the equation of motion $i\partial_t\psi + \Delta\psi/2m + \cdots = 0$. Consequently, the stable Fermi surfaces of dimension $d - p$ are now classified by groups $KR(\mathbf{R}^{p,1})$. These groups are periodic in p with periodicity 8; for low values of p, one finds $\mathbf{Z}$ for $p = 1, 5$, $\mathbf{Z}_2$ for $p = 2, 3$, and 0 for $p = 4, 6, 7, 8$. The ABS construction of the low-energy dispersion relation is again given by (7), and leads to coarse-grained relativistic Majorana fermions; the case of pseudo-Majorana fermions can be similarly incorporated.

The novel phenomenon for real fermions is the existence of Fermi surfaces with a $\mathbf{Z}_2$ charge. For example, in $2 + 1$ dimensions our framework predicts a stable Fermi line due to the $\mathbf{Z}$ charge in $KR(\mathbf{R}^{1,1})$, but also a stable Fermi *point* carrying a $\mathbf{Z}_2$ charge in $KR(\mathbf{R}^{2,1})$. The low-energy dispersion relation is that of a relativistic $SO(2, 1)$ Majorana fermion. Since the charge takes values in $\mathbf{Z}_2$, two such Fermi points when brought together would annihilate, and the system would form a gap. Such behavior has been observed [18]. Similarly, for real fermions in $d = 3$ we can have a Fermi surface carrying a $\mathbf{Z}$ charge, but also a Fermi line and a Fermi point, each supported by a $\mathbf{Z}_2$ charge in KR theory.

Having analyzed the stability of the Fermi surface in the ground state, one can extend our framework to include the classification of topological defects in Fermi liquids. In the semiclassical regime of small $\hbar$, one can consider slowly varying Fermi surfaces (in the spirit of [19]) and defects [20] as submanifolds in $(\mathbf{x}, t, \mathbf{k}, \omega)$. Repeating the analysis of this Letter will now lead to a classification of the spectrum of stable topological defects, and the dispersion relations of their low-energy modes, again in terms of K theory. This pattern is, indeed, very reminiscent of how K theory controls the spectrum of stable defects (in particular, D-branes) in string theory. It remains to be seen whether this analogy between Fermi liquids and string theory runs deeper than suggested by the results of this Letter.

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