# Asymptotic Evolution of Weakly Collisional Vlasov-Poisson Plasmas

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We study the role of (weak) numerical diffusion on the long time evolution of the Vlasov-Poisson plasma. We consider the classical problem of phase space vortex formation by particle trapping. We show that the asymptotic macroscopic state is not independent of diffusion even if the dissipative length scale is much shorter than any characteristic physical length scale of the system.

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The weakly collisional nature of many space and laboratory plasmas is a key point in theoretical modeling of plasma dynamics. These plasmas are typically rarefied and/or at high temperatures, so that the efficiency of collisions in bringing the particle distribution function (DF) close to a Maxwellian is reduced. The non-Maxwellian nature of the DF has been directly measured in the solar wind. In this case, the fluid description is inadequate and a kinetic description of the plasma must be introduced.

In the collisionless kinetic regime, the basic equation is the Vlasov equation self-consistently coupled to the electric and magnetic field equations. Concerning the role of collisions, in the electrostatic limit important results were obtained in the weakly collisional regime ($\nu_c \ll \omega_{\rm pe}$) where collective motions are unchanged, but nevertheless the role of collisions is essential since it strongly affects the free-streaming (ballistic) motion of the particles. This was first pointed out by Gould, O'Neil, Malmberg, and Wharton in the analytical and experimental work on plasma echoes generation by plasma wave pulses [1], where they suggested the echoes phenomenon as a tool for the study of collisional relaxation processes in plasmas. In the same years, Su and Oberman [2] demonstrated analytically, by using the simplified Fokker-Planck–type collision term of Lenard and Bernstein [3], that phase mixing and velocity phase space diffusion inducing collisional damping of free-streaming motion, can be so rapid both in space and time such as to prevent the possibility of generating plasma echoes. Later on, by using the same electron-electron collision operator, Skiff *et al.* [4] gave experimental and theoretical evidence on the existence of an intermediate regime where the spectrum of the low-frequency electrostatic ion waves is changed by collisions. In this context, recently Bhattacharjee *et al.* [5] have shown that in a weakly collisional plasma the Case–Van Kampen eigenmodes [6] of a collisionless plasma are replaced by a discrete spectrum (which form a complete set) giving the Landau damping results [7] in the limit of no collisions.

The theoretical study of collisionless plasma dynamics makes use, more and more, of large scale numerical simulations of the Vlasov equation with an Eulerian approach by using the so-called Vlasov codes (the Lagrangian approach, PIC codes, will be not discussed here). Starting from the pioneering studies of Ref. [8], the impressive development of computer power of these last years has allowed one to perform high resolution simulations in the electrostatic limit [9–11]. However, any long time numerical study of the Vlasov equation is unavoidably faced with the problem of small scales generation. As soon as the typical length scales of the fluctuations become comparable to the grid size, numerical dissipation comes into play leading the system to violate the conservative constraints of Hamiltonian dynamics and to reconnect close isolines of the DF. This process, formally forbidden, is well highlighted by the time evolution of the system invariants $N_i = \int f^i dxdv$, $i \geq 2$ and by the "entropy" $S = -\int f \ln(f) dxdv$ (here $f$ is the DF; note that $N_1$ is exactly conserved for the used algorithms), which show sudden variations when, for example, a "closed" vortex is formed in phase space as a consequence of electron trapping by the wave electric potential. To overcome this problem, the typical argument relies on the conjecture that if the dissipative scale is much smaller than any macroscopic physical length scale of the system, dissipation has no feedback on the macroscopic asymptotic evolution of the system.

The role of dissipative effects on the asymptotic evolution of a collisionless system is related to the existence of equilibria solutions corresponding to different energy states. In the case of a Vlasov-Poisson 1D system, it is known that a "large" set of Bernstein-Greene-Kruskal (BGK) stationary solutions [12] exist corresponding to the formation of phase space vortices of different size and different mean values of the electric field amplitude. The dynamical transition from an initial Maxwellian to a final BGK-type state (hereafter just BGK), can be justified by saying that the closed isolines are a coarse graining view of the DF and that the energy difference between the initial and the final states is stored on the very small DF oscillations. After the formation of the BGK structure, the evolution is again collisionless up to any new effect trying to destabilize the BGK state. Since the dissipative mechanism is responsible for the closure of the DF lines leading to the formation of vortex structures, dissipation influences the number and distribution of trapped particles in the vortex [13] and so the asymptotic macroscopic state of the system, even if the diffusive length scale is the smallest characteristic length scale at play.

We consider the standard problem of ''nonlinear Landau damping'' (NLD) [14] in the regime where the bounce time $\tau_B$ is comparable to the Landau time $\tau_L$ (see Ref. [15] and references therein). In this case particle trapping stops the Landau wave-particle collisionless damping [16] leading to the formation of phase space electron holes which have been identified as finite superposition of traveling BGK waves [17]. For initial amplitudes larger than a critical value $\epsilon^*$ separating the damping (or Landau) and the non-damping (or O'Neil) regimes [18], the asymptotic state is characterized by a constant electric field amplitude oscillating around a mean value with a characteristic period $\tau_{LF}$ much longer than the trapping time [9,10]. The long time, low-frequency oscillating regime has been very recently analyzed in detail by Valentini *et al.* [19] using the Lagrangian trajectories of particles in the resonant region. They showed two main electron populations: the chaotic one, located nearby the separatrix characterized by phase space flightslike trajectories, and the trapped population performing a nonergodic dynamics. They conclude that the complex interplay between these two populations is responsible for the asymptotic low-frequency oscillating evolution as a consequence of a ''trapping-detrapping'' mechanism; furthermore, they claimed that the (most probable) flight duration is connected with the low-frequency period, $\tau_{LF} = 120$, observed in the simulation. However, we see here that diffusive effects, not considered in Ref. [19], are crucial to determine the number and the phase space distribution of the resonant (and nearby) particles and so to estimate $\tau_{LF}$. The NLD problem has also recently been investigated in the paper by Ivanov *et al.* [20], where the study on the transition between the damping and nondamping regimes has been revised with a phase transition approach. Reference [20] is also based on a very fine analysis of long time Vlasov-Poisson numerical simulations, but again the possible role of dissipative effects has been neglected.

By using a II order (VL2) or a III order (VL3) Vlasov code [21], which exactly conserves the total charge, we integrate the 1D-1V Vlasov-Poisson system of equations (normalized to electron characteristic quantities):

$$\partial f_a/\partial t + v\partial f_a/\partial x - \mu\partial\phi/\partial x\partial f_a/\partial v = 0; \quad a = e, p,$$

$$\partial^2\phi/\partial x^2 = \int (f_e - f_p)dv; \quad E = -\partial\phi\partial x,$$

$$f_e(x, v, 0) = (1/(\sqrt{2\pi}v_{\rm th})e^{-(v^2/2v_{\rm th}^2)}[1 + \epsilon\cos(kx)],$$

with $\epsilon = 0.05$, $k\lambda_D = 0.4$, $\mu = \pm m_e/m_a$, $-5 \leq v/v_{\rm th} \leq 5$, and $L_x = 5\pi\lambda_D$. Periodic boundary conditions are used. The dimension of the numerical box is chosen in order to have stable vortices since for $L_x > 2\pi/k$ the sideband instability [22] could develop in some circumstances [10]. In the following, we present the results of one VL2 run (A) and three VL3 runs (B, C, D), all starting with the same initial conditions, but from A to D with decreasing numerical diffusion; we use $N_x = 128$, $N_v = 701$ mesh points in runs A and B, $N_x = 256$, $N_v = 1401$ in run C, $N_x = 512$, $N_v = 2801$ in run D. Roughly speaking, the numerical diffusive length scales are of the order of the mesh size, $0.03 \leq dx \leq 0.12$, $0.0035 \leq dv \leq 0.014$, while the phase space characteristic scales are the Debye length, $\lambda_D = 1$, the electron thermal velocity $v_{{\rm th},e} = 1$, the perturbation wavelength, $\lambda = 5\pi\lambda_D$, the vortex dimensions, $l_x \simeq 12\lambda_D$, $l_v \simeq 1v_{{\rm th},e}$. To summarize, run A is the most dissipative (remember that the VL2 algorithm is more diffusive than VL3), and run D is the least dissipative. The numerical recurrence time [23], $T_{\rm rec} = 2\pi/(k\Delta v)$, is always (much) longer than the final time of the simulation. In Fig. 1, first frame, we show the third invariant $N_3$ vs time. In all cases, the invariants are constant in the initial and asymptotic phase of the evolution (except for run A which continues slowly to decrease asymptotically). In correspondence with the vortices' formation, a sudden jump is observed which is delayed more and more as dissipation is reduced. The asymptotic value of the invariants decreases as dissipation increases. This behavior is typical of Hamiltonian plasma simulations, as, for example, in the case of collisionless (fluid) magnetic reconnection in correspondence with magnetic islands formation. In Fig. 1, second frame, we show the time evolution of the entropy $S$. We again observe a strong variation of $S$ as soon as the DF isolines start to diffuse, later for the less dissipative case,

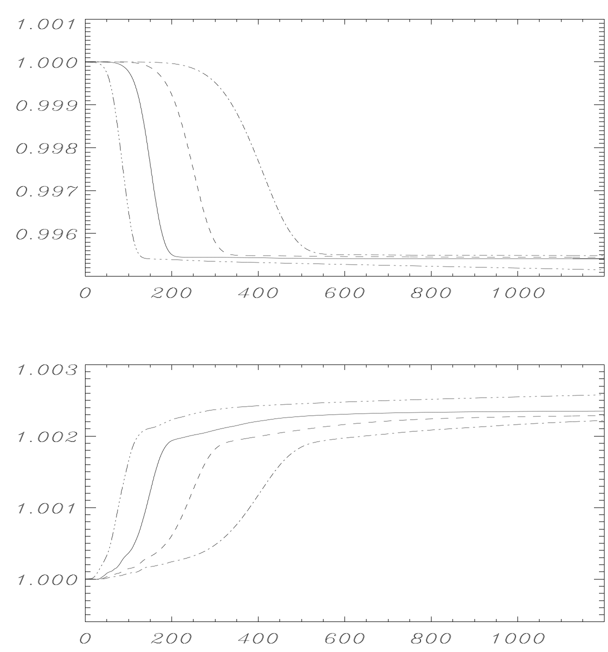


FIG. 1. First frame: the time evolution of the third invariant $N_3$. The three dot-dashed, continuous, dashed, and dot-dashed lines correspond to runs A to D, respectively. Second frame: the time evolution of the entropy (same style).

run D, and $S$ again becomes nearly constant in the asymptotic limit. We emphasize that the entropy difference between runs A and D at the final time, $\delta S_{\mathrm{A,D}} = S_{\mathrm{A}} - S_{\mathrm{D}}$, is comparable to the total entropy variation $\Delta S_{\mathrm{D}}$ of run D (i.e., between the initial and the final times): $\delta S_{\mathrm{A,D}} \simeq 0.23 \Delta S_{\mathrm{D}}$. In Fig. 2 we show, for runs B and D, the phase space vortices (the same contour levels) centered around the resonant velocity. The vortices are characterized by finer and finer ripples as diffusion is reduced. We have calculated the number of particles contained in each vortex by defining the separatrix as the largest closed contour level. Inside each separatrix we have integrated the distribution function and normalized to one the total number of trapped particles of run D, $n_{\mathrm{D}}^{\mathrm{vort}} = 1$. The maximum numerical error due to different grid resolution is estimated of the order of $\sim 1\%$. We found $n_{\mathrm{C}}^{\mathrm{vort}} = 0.87$, $n_{\mathrm{B}}^{\mathrm{vort}} = 0.83$, and $n_{\mathrm{A}}^{\mathrm{vort}} = 0.57$. In Fig. 3 we show two plots of the DF at $t = 650$ in the resonant region at $v = 3.14$ vs $x$ and at $x = 7.85$ vs $v$. The thicker continuous line and the continuous line refer to run B and run D, respectively. The dashed and dash-dotted lines are obtained after smoothing (with a Fourier or with a finite differences technique) the run D vortex on a phase space cell size $dx \times dv$ comparable to that of run B. We see that the strong gradients are somewhat reduced by the smoothing, but that the (dashed) smoothed lines are still very different from that of run B corresponding to a significant different distribution of the trapped particles in the vortices. Therefore, the run B vortex cannot be considered as a coarse grained view of the run D vortex. This difference is even more evident when compared with the vortex of run A. We conclude that vortices obtained with a different diffusion coefficient do not commute by *a posteriori* diffusive smoothing. The signature of the difference between the asymptotic states can be seen on the electric field amplitude evolution shown in Fig. 4 (here only runs B and D are shown) where we eliminated the high frequency oscillations. We found $\tau_{\mathrm{A}}^{lf} \simeq 46$, $\tau_{\mathrm{B}}^{lf} \simeq 40$, $\tau_{\mathrm{C}}^{lf} \simeq 67$, and $\tau_{\mathrm{D}}^{lf} \simeq 75$. We note that in Ref. , with a VL2 scheme and $N_x = 512$, $N_v = 1601$ (all other parameters equal), they get $\tau_{\mathrm{LF}} \simeq 120$. By performing other VL2 and VL3 simulations (not reported here), the trend of an increasing low-frequency period with increasing resolution is observed for both the VL2 and the VL3 algorithms. This trend is in agreement with the trapping-detrapping mechanism proposed in Ref. [19] since increasing the resolution corresponds to decreasing the possibilities of diffusing around the separatrix (i.e., the system becomes more and more collisionless). Therefore, in the limit of no dissipation (infinite resolution) the electric field amplitude should eventually be constant. As a consequence, the NLD problem can be considered as a fine numerical test for long time simulations with kinetic codes. Finally, we observe that the min or max value of the electric field amplitude in runs A to D are significantly different.

In summary, we have presented numerical evidence on the importance of dissipative effects in a Vlasov-Poisson

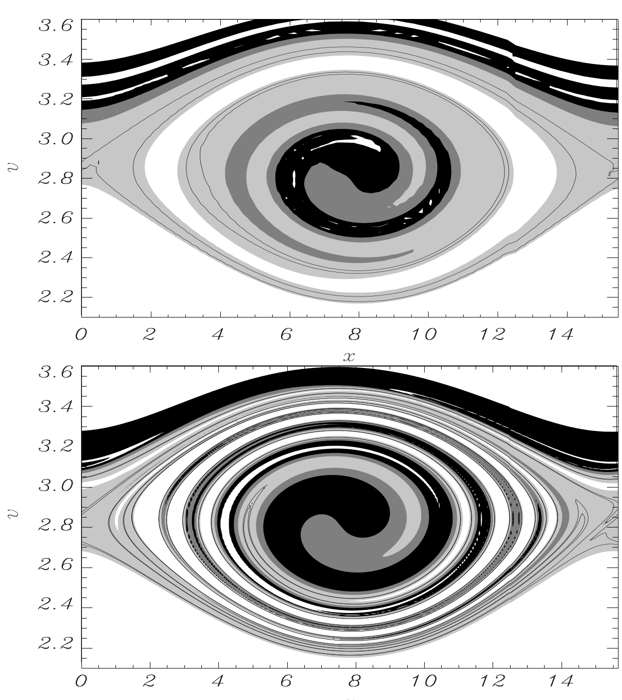


FIG. 2. The shaded isocontour of the DF of runs B and D in a strip of the phase space centered around $v_{\mathrm{res}} \simeq 2.9$.

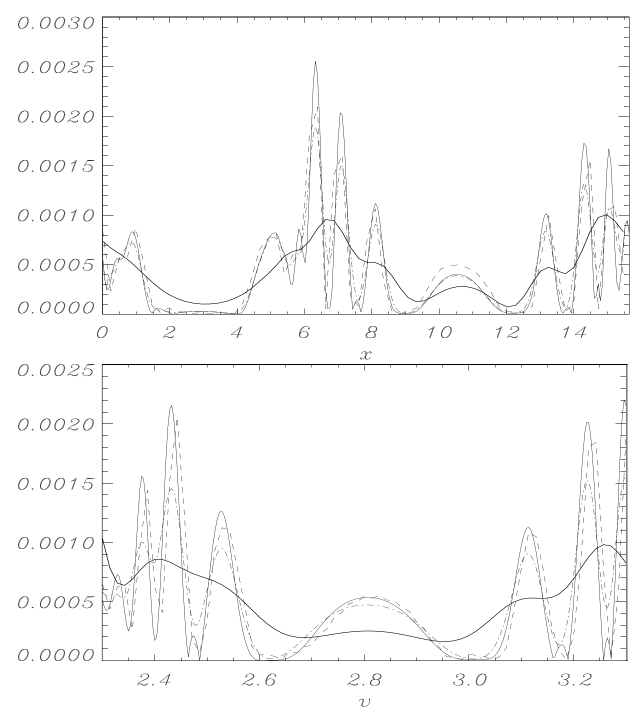


FIG. 3. The DF ($t = 650$) at $v = 3.14$ vs $x$, first frame, and at $x = 7.85$ vs $v$, second frame, for run B (continuous thick line), run D (continuous line), run D smoothed with Fourier and with finite differences, dashed and dot-dashed lines, respectively.

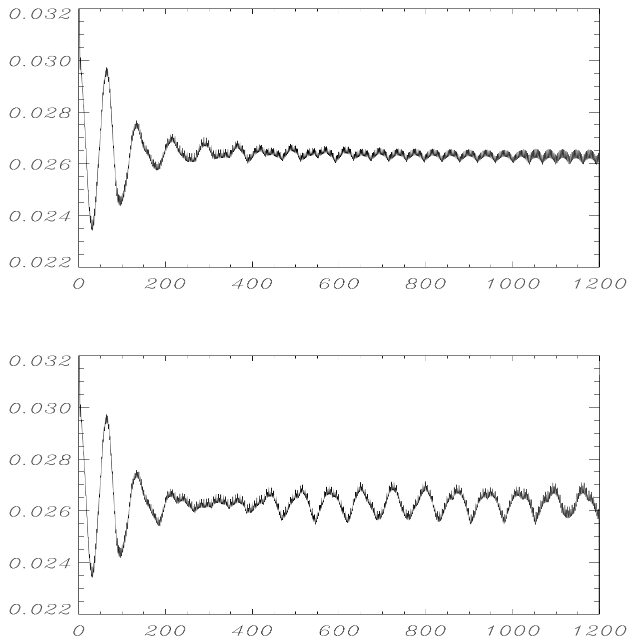


FIG. 4. The low-frequency time evolution of the electric field amplitude, runs B and D.

plasma. In our study dissipative effects are given by the computational discrete mesh and by the choice of the numerical algorithm. Our results are in qualitative agreement with those of Ref. [5] in the sense that, even if the collective dynamics is unchanged in the presence of weak collisions, nevertheless the variation on the small scale fluctuations can have important feedback on the asymptotic large scale dynamics. In agreement with Ref. [13], we found that the closure of the DF lines driven by dissipative effects has a qualitative and quantitative influence on the distribution of the trapped particles and so on the asymptotic BGK-like state of the system. This is due to the lack in the Vlasov-Poisson system of a microscopic length scale controlling the physics between the macroscopic structures and the dissipative scale. A different behavior is found when considering the collisionless magnetic reconnection problem driven by electron inertia (with $l_{\rm diff} \ll d_e$). In that case, at least in a regime where the equations are mathematically similar to the Vlasov-Poisson system [24], we found [25] that long time magnetic island pulsations are independent of the grid resolution (at least in the limit of computer resources) due to the inertial length scale which controls the microphysics of the reconnecting layer. Finally, we note that, even if numerical diffusion plays a role in the macroscopic long time behavior of the system, our results confirm that there are no indications of wave amplitude decay in the long time nonlinear evolution (proposed in Ref. [26]), as recently pointed out analytically [18,27] and numerically [9].

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