Repulsive Synchronization in an Array of Phase Oscillators

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We study the dynamics of a repulsively coupled array of phase oscillators. For an array of globally coupled identical oscillators, repulsive coupling results in a family of synchronized regimes characterized by zero mean field. If the number of oscillators is sufficiently large, phase locking among oscillators is destroyed, independently of the coupling strength, when the oscillators' natural frequencies are not the same. In locally coupled networks, however, phase locking occurs even for nonidentical oscillators when the coupling strength is sufficiently strong.

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Introduction.—Interest in the dynamics of arrays of coupled nonlinear oscillators is driven by important applications in physics (coupled Josephson junctions [1], laser arrays [2]), engineering (phased antenna arrays [3,4]), biology (neural networks [5,6]), etc. Locally coupled oscillators can exhibit a variety of complex spatially nonuniform regimes including clustering, wave propagation, and chaos. The most interesting phenomenon occurring in a globally coupled array of oscillators with distributed frequencies is the transition to a globally synchronized regime when the coupling strength between the oscillators is increased. It can be described analytically with the Kuramoto model of coupled phase oscillators [7–9]. Subsequently, many extensions of the original Kuramoto model have been studied, including more general, time-delayed, spatially nonuniform, adaptive coupling, noise, external driving, etc. (see, e.g., [10]). We should note that the coupling among the oscillators has almost exclusively been assumed to be *attractive*; i.e., two coupled oscillators tend to oscillate in phase. However, inhibitory (or repulsive) coupling is very common in biological systems. In Ref. [11] it was shown that sparse long-range inhibitory coupling in addition to local excitatory coupling can produce stable nonuniform phase distributions in an array of identical phase oscillators. Kim *et al.* [12] studied pattern formation in a two-dimensional array of identical phase oscillators with phase-shifted coupling, which for the phase shift π corresponds to the repulsive coupling.

In this Letter we will analyze the dynamics of phase oscillators with *purely repulsive* coupling. Obviously, two repulsively coupled oscillators tend to oscillate in antiphase (see, e.g., [12,13]). The situation is less trivial for a large number of oscillators. We show that the dynamics of an array of globally coupled oscillators with identical natural frequencies settle onto one of several possible synchronized regimes characterized by zero mean field [14]. This result changes drastically when the oscillators have nonidentical natural frequencies. A small number of oscillators will still synchronize for sufficiently strong coupling; however, a large ensemble of oscillators ($N > 3$) will not synchronize for *any* coupling strength. We also consider the dynamics of a locally coupled one-dimensional array of oscillators (when the span of interaction is less than the system size). In this case, for sufficiently strong repulsive coupling, even an array of oscillators with nonidentical frequencies will synchronize with an approximately linear phase distribution. We expect similar behavior to be observed more generally in repulsively coupled physical and biological systems.

Global coupling.—Let us first consider the dynamics of a globally coupled array of phase oscillators known as the Kuramoto model [7]:

$$\dot{\varphi}_j = \omega_j - \frac{\kappa}{N}\sum_{n=1}^{N}\sin(\varphi_n - \varphi_j). \tag{1}$$

Without loss of generality, we assume that the mean natural frequency of the oscillators is zero, $N^{-1}\Sigma_{j=1}^{N}\omega_j = 0$. Note the negative sign in front of the coupling term corresponds to the case of repulsive coupling which is the focus of this Letter. We can introduce the complex mean field

$$R = re^{i\psi} = \frac{1}{N}\sum_{n=1}^{N} e^{i\varphi_n}, \tag{2}$$

so Eq. (1) can be written as

$$\dot{\varphi}_j = \omega_j - \kappa r\sin(\psi - \varphi_j). \tag{3}$$

Numerical simulations of Eq. (1) show that when the natural frequencies of the oscillators are identical ($\omega_j = 0$), the mean field always reaches zero after an initial transient period. This stationary regime corresponds to all of the oscillators having different phases. However, the phase distribution is neither uniform nor unique: the stationary regime can exhibit an arbitrary phase distribution among individual oscillators, subject only to the constraint $R = 0$. All these solutions are neutrally stable.

Indeed, it can be shown that the Jacobian of the linearized Eq. (1) has $N - 2$ zero eigenvalues and two negative eigenvalues corresponding to the decaying magnitude of the mean field R. It can also be shown that the zero-mean-field solution is globally stable [15].

For nonidentical frequencies, the dynamics of the array are more complicated. The mean field oscillates for small values of coupling κ; however, for larger coupling strengths, the oscillators may synchronize. The transition to the synchronized regime depends strongly on the number of oscillators in the array.

For two oscillators with frequencies $\pm\omega_0$, symmetry dictates that the mean field does not oscillate in the synchronized regime, and that the phases of the oscillators are symmetric with respect to the mean field phase $\psi = 0$: $\varphi_1 = -\varphi_2 = \varphi$. The equation for the single phase φ

$$\dot{\varphi} = \omega_0 + \frac{\kappa}{2}\sin 2\varphi \tag{4}$$

has two stationary solutions: $-\arcsin(2\omega_0\kappa^{-1})/2$ and $-\pi/2 + \arcsin(2\omega_0\kappa^{-1})/2$. The first solution is unstable, but the second solution is stable. These stable and unstable synchronized solutions merge and disappear when $\kappa\omega_0^{-1} < 2$, giving rise to the nonsynchronized regime. In the synchronized regime, the mean field decreases with the coupling strength as

$$R = \sqrt{\frac{1 - \sqrt{1 - 4\omega_0^2/\kappa^2}}{2}}. \tag{5}$$

This solution agrees with the numerically computed bifurcation diagram shown in Fig. 1(a).

For the case of three oscillators with a symmetric distribution of frequencies $\omega_{1,2,3} = -\omega_0, 0, \omega_0$, we introduce the phase differences $\Phi_0 = \varphi_2 - \varphi_1$, $\Phi_1 = \varphi_3 - \varphi_2$. The equations for $\Phi_{0,1}$ are

$$\dot{\Phi}_i = \omega_0 + \frac{\kappa}{3}[\sin(\Phi_i + \Phi_{1-i}) - \sin\Phi_{1-i} + 2\sin\Phi_i], \tag{6}$$

$i = 0, 1$. The symmetric synchronized solution corresponds to the fixed point $\Phi_0 = \Phi_1 = \Phi$, which is defined by the transcendental equation

$$\omega_0 = -\frac{\kappa}{3}[\sin\Phi + \sin 2\Phi]. \tag{7}$$

This equation has two solutions which merge and disappear, via a saddle-node bifurcation, for $\kappa\omega_0^{-1} > 96[(3 + \sqrt{33})\sqrt{30 + 2\sqrt{33}}]^{-1} \approx 1.704$. However, unlike the case of two oscillators, this condition determines the existence but not the stability of the synchronized solution. To determine the conditions for the stability of the synchronized solution, we linearize Eqs. (6) near the fixed point (7) and find the two eigenvalues $\lambda_1 = \kappa\cos\Phi$, $\lambda_2 = \frac{\kappa}{3}(2\cos 2\Phi + \cos\Phi)$. It is easy to check that both these eigenvalues are negative if $\arccos[(1 + \sqrt{33})/8] < |\Phi + \pi| < \pi/2$, which according to (7) corresponds to a stability threshold $\kappa = 3\omega_0$ [cf. Fig. 1(b)].

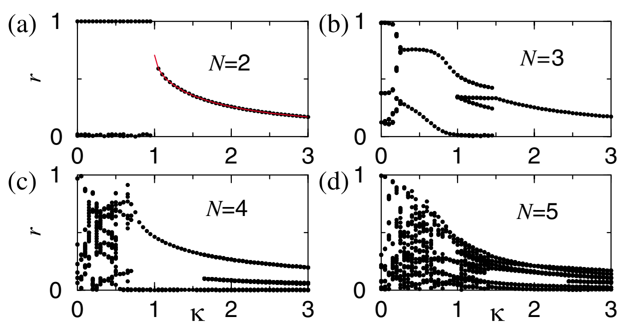

FIG. 1 (color online). Bifurcation diagrams: extrema of the mean field magnitude r vs κ for $N = 2$, 3, 4, and 5 oscillators with natural frequencies uniformly spread across the interval $[-0.5, 0.5]$. Solid line in panel $N = 2$ corresponds to the analytical formula (5).

Now we examine the dynamics of an ensemble of many oscillators with distributed frequencies. Without loss of generality, we assume that they are distributed within the interval $[-\omega_0, \omega_0]$. For large N, the synchronized solution must be unstable. Indeed, according to Eq. (3), if there is a completely synchronized solution, the magnitude of the corresponding mean field r must satisfy the condition $\kappa r > \omega_0$. In the reference frame corotating with mean field, the synchronized solution of (3) is stationary. In the limit of large N, Eq. (3) in the linear approximation decouples because perturbing the phase of one of the oscillators without changing phases of other oscillators will only affect the mean field by a negligible amount $O(1/N)$. Therefore, the eigenvalue corresponding to the jth oscillator is $\lambda_j = \kappa r\cos\phi_j^0 + O(1/N)$ (where ϕ_j^0 is the stationary phase of jth oscillator with respect to the phase mean field). The phases of at least some of the oscillators must lie within the interval $[-\pi/2, \pi/2]$. Therefore, the corresponding eigenvalues of the synchronized solution are positive, so the synchronized solution is unstable. In fact, our numerical simulations show that the synchronized solution is unstable for any κ and for all $N \geq 4$ (see, e.g., bifurcation diagrams for $N = 4$, 5 in Figs. 1(c) and 1(d); however, we were not able to prove this analytically for arbitrary N and frequency distributions.

In the unsynchronized regime, the oscillators maintain their natural frequencies *on average*; however, due to coupling they adjust their phases so as to minimize the mean field. The equation for the mean field for nonidentical frequencies can be written as

$$\dot{R} = \frac{1}{N}\sum_{n=1}^{N}\omega_n e^{i\varphi_n} - \frac{\kappa}{2}\left[R - R^*\frac{1}{N}\sum_{n=1}^{N}e^{2i\varphi_n}\right]. \tag{8}$$

In the limit of large N, the last term in the right-hand side can be neglected, and the first term can be approximated as $N^{-1}\sum_{n=1}^{N}\omega_n e^{i(\omega_n t+\xi_n)n}$, where ξ_n is the random phase of an individual oscillator [16]. The resulting equation

$$\dot{R} = \frac{1}{N}\sum_{n=1}^{N}\omega_n e^{i(\omega_n t+\xi_n)} - \frac{\kappa}{2}R \tag{9}$$

can be easily solved. For large κ, the mean field can be written as $R(t) = 2(\kappa N)^{-1}\sum_{n=1}^{N}\omega_n e^{i(\omega_n t+\xi_n)}$. Assuming a uniform distribution of frequencies in the interval $[-\omega_0, \omega_0]$, we get the standard deviation of $R(t)$,

$\sigma_r = 2\omega_0(3N\kappa^2)^{-1/2}$, which agrees with numerical simulations for large N (see Fig. 2).

Local coupling.—Now we consider the case of a one-dimensional array of phase oscillators repulsively coupled to their L left and L right neighbors, where $L < N/2$,

$$\dot{\varphi}_j = \omega_j - \frac{\kappa}{2L+1}\sum_{n=-L}^{L}\sin(\varphi_{j-n} - \varphi_j). \tag{10}$$

We assume periodic boundary conditions: $\varphi_{j+N} = \varphi_j$. The limiting case of $L = 1$ corresponds to the nearest-neighbor coupling. It is well known that for repulsive nearest-neighbor coupling an antiphase synchronization in the form of π oscillations emerges (see, for example, [17]). Here we address the general case of $L \geq 0$.

Our numerical simulations show that for $L < N/2$ and large enough κ, oscillators with frequencies distributed within a finite range $[-\omega_0, \omega_0]$ converge to an attractor with an almost *linear* phase distribution $\varphi_j = \Delta\varphi j$. The larger κ, the closer the phase distribution is to a linear profile; see Fig. 3(a) and 3(c). For the case of identical oscillators ($\omega_0 = 0$) and an arbitrary nonzero value of coupling, the phase distribution is exactly linear. To satisfy the periodic boundary conditions, the value of the phase shift $\Delta\varphi$ may only take discrete values $\Delta\varphi_n = \pm 2n\pi/N$, $n = 1, 2, 3, \ldots$ [18]. The selected phase shift $\Delta\varphi_s$ depends on both L and N, but is independent of the coupling constant κ. For a large system ($N \gg L$), the phase shift between the neighboring oscillators approaches a certain value $\Delta\varphi_\infty$, which depends on L only [Fig. 4(d) shows the selected phase difference $\Delta\varphi_s$ as a function of L for an array of 99 identical oscillators]. Note that similar domains with linear phase distributions were found in a two-dimensional array of phase oscillators with finite-range phase-shifted coupling [12].

Analytical understanding of this phenomenology can be gained by noticing that model (10) can be obtained from the variational principle $\dot{\varphi}_j = -\partial_{\varphi_j}F$, where the free energy is

$$F = -\sum_{j=1}^{N}\omega_j\varphi_j + \frac{\kappa}{2(2L+1)}\sum_{j=1}^{N}\sum_{n=-L}^{L}\cos(\varphi_{j-n} - \varphi_j), \tag{11}$$

see, for example, [5]. Stable stationary solutions of (10) correspond to local minima of F. Assuming $\omega_j = 0$ and a linear phase grid $\varphi_j = j\Delta\varphi$, we obtain from (11) the free energy density

$$f \equiv \frac{F}{N} = \frac{\kappa\sin[\Delta\varphi(L+1)] + \sin(\Delta\varphi L)}{2(2L+1)\sin(\Delta\varphi)}. \tag{12}$$

For a fixed L, this free energy as a function of $\Delta\varphi$ has multiple minima [Figs. 4(a) and 4(c)]. The most stable solution corresponds to the global minimum of f. For $N \gg L \gg 1$, $f \approx \kappa\sin(\Delta\varphi L)/4\Delta\varphi L$, and its global minimum $\Delta\varphi_m$ is given by $\Delta\varphi_m L \approx 4.49$. For finite L and large N, $\Delta\varphi_m L$ is not constant, but depends on L. For $L \sim N$, as was mentioned before, the selected phase shift $\Delta\varphi_s$ must be taken from a discrete set $\Delta\varphi_n$ [corresponding values of free energy f_n are shown in Fig. 4(b)]. This leads to a stepwise dependence of $\Delta\varphi_s$ on L for a fixed N [see Fig. 4(d)]. As seen in Fig. 4(c), for $2L + 1 \to N$ the phase difference approaches $4\pi/N$. At the same time, $f_n \to 0$.

The mechanism providing stability of the linear phase grid is easily understood. Note that the free energy density f is equal to the local "mean field" of the jth oscillator in its corotating reference frame $f = R_j = (2L+1)^{-1}\times \sum_{n=-L}^{L}\exp[i(\varphi_{j-n} - \varphi_j)]$. For the resulting values of $\Delta\varphi_m$, the free energy is negative ($f_m < 0$); i.e., the local mean field is in antiphase with the oscillator.

For nonidentical oscillators with frequencies distributed within a finite interval $[-\omega_0, \omega_0]$, the situation is qualitatively the same. The oscillators synchronize to the local "mean field" $R = f_n$ with the phases approximately forming a linear grid. Since the local mean field is finite and negative, the oscillators remain synchronized as long as all $|\omega_j| < \kappa|R_j|$. However, as we have just seen, $f_n \to 0$ as $2L + 1 \to N$; i.e., as the span of interactions approaches the system size, the local mean field disappears. At the value of L for which $f_n < \omega_0/\kappa$, synchronization is lost.

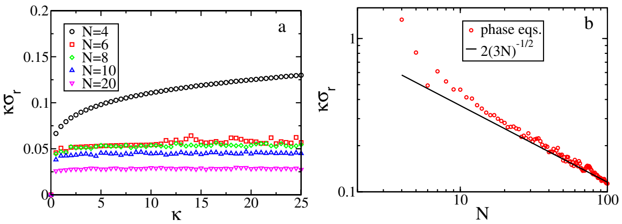

FIG. 2 (color online). The normalized standard deviation of the mean field $\kappa\sigma_r$ vs κ for several values of N corresponding to the nonsynchronized regime (a) and vs the number of oscillators for large coupling, $\kappa = 5$ (b) for $\omega_0 = 1$. The solid line in (b) corresponds to the theoretical solution.

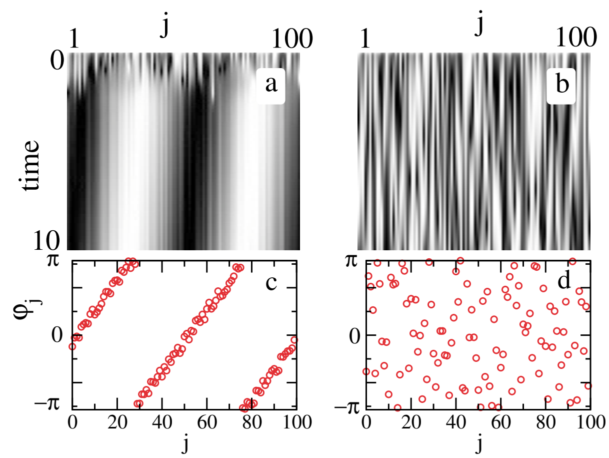

FIG. 3 (color online). (a), (b) Space-time plots of $\sin[\varphi_j(t)]$ for $N = 100$, $\kappa = 2$, and $\omega_0 = 0.1$ in the synchronized regime $L = 40$ (a) and unsynchronized regime $L = 48$ (b). (c), (d) Phase distributions for oscillators at $t = 1000$. Gray scale from white to black corresponds to the range $[-1, 1]$.

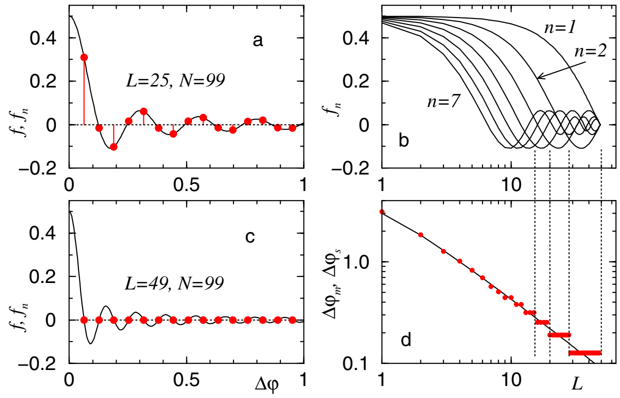

FIG. 4 (color online). Dependence of the free energy density f (solid line) and f_n (closed circles) on $\Delta\varphi$ for two values of L according to (12) (a), (c). (b) f_n vs L for several values of $\Delta\varphi_n$. (d) Dependence of $\Delta\varphi_m$ (line) and $\Delta\varphi_s$ (symbols) on L for $N = 99$ identical oscillators.

This is confirmed by direct numerical simulation of Eq. (10) [see Figs. 3(b) and 3(d)].

Summary.—In this Letter we investigated the dynamics of repulsively coupled phase oscillators. With global coupling, the system dynamics converge to a regime that minimizes the mean field. For identical frequencies, the mean field decays to zero for any nonzero coupling coefficient, whereas for nonidentical frequencies, the mean field remains finite for any finite value of coupling strength. Furthermore, for the number of oscillators $N > 3$, repulsive coupling fails to synchronize the array, and the mean field fluctuates for any value of coupling constant. A one-dimensional array with local coupling spanning a finite number of neighbor oscillators synchronizes to a locally linear grid of phases when the span of coupling is sufficiently smaller than the system size.

The model of repulsive coupling studied in this Letter is rather idealized. However, it may serve as a paradigm for many biological networks in which different elements compete against each other. The best known example of such networks are neuronal ensembles with inhibitory coupling [5]. It is well known that in such systems, nonuniform synchronized oscillation patterns may emerge. Such neuronal circuits underlie central pattern generators in many biological systems [6].

In this Letter we considered only two simple cases of repulsively coupled oscillators: a globally coupled array and a one-dimensional array with local connections. The influence of the network architecture on the dynamics of repulsively coupled oscillators is an interesting open question. We also assumed that the coupling function (which has to be 2π periodic) has only fundamental Fourier component. It is well known that including higher harmonics in the coupling function can lead to different behavior near the synchronization transition in a system of attractively coupled oscillators [9,19]. Based on our preliminary numerical studies, we believe that this is not the case for repulsive synchronization; however, this issue deserves further investigation. Another interesting issue left open is the effect of noise on the dynamics of repulsively coupled oscillators. For the case of many identical oscillators, the continuum of stationary states with zero mean field suggests that the noise will cause the system to drift diffusively among these states.

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[14] Here we define synchronization broadly as a regime in which oscillators maintain fixed phase relationships.
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