Strange-Quark Mass from Tau-Lepton Decays with $O(\alpha_s^3)$ Accuracy

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The first complete calculation of the quadratic quark mass correction to the correlator of the two currents relevant for the strangeness-changing semihadronic tau-decay rate is presented *including its real* part at the four-loop level. This allows us to perform the extraction of the strange-quark mass m_s from the decay width of the tau-lepton with *full* $O(\alpha_s^3)$ accuracy. In agreement with previous estimates, the newly computed α_s^3 term proves to be rather large. This justifies inclusion of the similarly estimated α_s^4 term in phenomenological analysis. Combined with an updated value of $V_{\rm us}$ and an "improved" version of the renormalization group improvement of the perturbative series, this leads to an increase of the central value of m_s by about 20% and a partial reduction of the theoretical uncertainty by about 50%.

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Introduction.—The investigation of Cabibbo-suppressed semileptonic τ decays has developed into one of the important themes of τ-lepton physics. Combined theoretical and experimental studies provide an independent and fairly precise value for the strange-quark mass m_s and may in the future also lead to a competitive determination of the Cabibbo-Kabayashi-Maskawa matrix element $V_{\rm us}$ [1–9]. The decay proceeds into vector and axial-vector current induced final states, which can be further separated experimentally into the spin zero and spin one components [10]. In addition to the total rate also various moments of the distribution in s, the invariant mass of the hadronic state, can be considered and is discussed below.

The theory prediction for the rate and moments is based on perturbative QCD with small contributions from nonperturbative condensates [11]. The calculation is greatly simplified by the smallness of the strange-quark mass which justifies an expansion in powers of m_s^2/M_τ^2. Up to now the m_s dependence of the total rate was known only up to order α_s^2. To push the precision further, estimates for the third order coefficient were used which were based on the principle of minimal sensitivity (PMS) [12] or fastest apparent convergence (FAC) [13]. In this Letter the exact result for this coefficient is presented. It is deduced from the first complete calculation of the finite part of a specific correlator in four-loop approximation. This calculation is based on the conceptual developments described in [14–16] and requires extensive use of computer algebra [17]. The result confirms the PMS-FAC estimate and justifies the use of the same approach for an estimate of the α_s^4 coefficient. Inclusion of this next term leads to a decrease of the central value of m_s by a few MeV and a partial reduction of the theoretical uncertainty by 50%.

The phenomenological analysis is based on the, by now standard, approach of contour improved perturbation theory (CIPT) [18,19]. Two variants are discussed. We argue in favor of one of them which leads in total to more stable results. The newest developments for $V_{\rm us}$ are included in the analysis.

The main outcome of the study of the total hadronic τ width (normalized to the known leptonic width)

$$R_\tau = \frac{\Gamma(\tau \to \text{hadrons} + \nu_\tau)}{\Gamma(\tau \to l + \bar{\nu}_l + \nu_\tau)} \tag{1}$$

has been a remarkable confirmation of perturbative QCD. It was found that the value of the strong coupling constant α_s as obtained from R_τ is in good agreement with those obtained from completely different experiments such as the Z boson decay into hadrons [20–23]. Furthermore, the strangeness-changing (Cabibbo-suppressed) part $R_{\rm S}$ of the decomposition of the decay rate and the moments [19]

$$\begin{aligned} &R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds} = R_{\tau,\rm NS}^{kl} + R_{\tau,\rm S}^{kl}, \\ &R_\tau^{00} \equiv R_\tau, \qquad R_{\tau,\rm NS}^{00} \equiv R_{\tau,\rm NS}, \qquad R_{\tau,\rm S}^{00} \equiv R_{\tau,\rm S} \end{aligned} \tag{2}$$

into nonstrange (NS) and strange (S) components can be used to determine the strange-quark mass, one of the fundamental parameters of the standard model. The moments as introduced above can be experimentally determined from the measured distribution in the invariant mass of the final state hadrons. Hadronic physics is encoded in the quantities r_{ij}^{kl} defined through

$$R_{\tau,\rm NS} + R_{\tau,\rm S} = 3 S_{\rm EW}(|V_{\rm ud}|^2 r_{\rm ud} + |V_{\rm us}|^2 r_{\rm us}), \tag{3}$$

where v_{ij} stands for the CKM matrix and $S_{\rm EW}$ for the universal electroweak correction [24,25]. The functions r_{ij}^{kl} are directly related to the correlator of the charged current $j_\mu(x) = \bar{\psi}_i \gamma_\mu (1 - \gamma_5) \psi_j$

$$i\int dx e^{iqx}\langle T[j_\mu(x)j_\nu^\dagger(0)]\rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi_{ij}^{(q)}(q^2) + g_{\mu\nu}q^2\Pi_{ij}^{(0)}, \quad (4)$$

through

$$r_{ij}^{k,l} = r_{ij}^{(q)k,l} + r_{ij}^{(0)k,l} = 2i\pi\oint_{|s|=M_\tau^2}\frac{ds}{M_\tau^2}[w^{(q)k,l}(x)\Pi_{ij}^{(q)}(s) + w^{(0)k,l}(x)\Pi_{ij}^{(0)}(s)], \quad (5)$$

where $x = s/M_\tau^2$, and the weight functions $w^{(q)k,l} = (1 + 2x)(1-x)^{k+2}x^l$, $w^{(0)k,l} = -2x^{l+1}(1-x)^{k+2}$. The behavior of the perturbative series for the part of the integral arising from $\Pi^{(q)}$ is more stable than the one from $\Pi^{(0)}$ [26]. However, the latter can either be modeled theoretically and phenomenologically on the basis of scalar and pseudoscalar resonance physics [4,27–29], or, being solely determined by spin zero contribution in the τ decays, be determined experimentally through the analysis of angular distribution of the $K\pi$ and $K\pi\pi$, thus separating spin zero and spin one contributions [10]. Restricting to scalar and pseudoscalar channels one finds

$$R_{\tau,S}^{kl(J=0)} = \int_0^{M_\tau^2} ds\left(1-\frac{s}{M_\tau^2}\right)^k\left(\frac{s}{M_\tau^2}\right)^l\frac{dR^{J=0}}{ds} = \frac{-3}{2}S_{EW}|V_{us}|^2 r_{us}^{(0)k,l-1}. \quad (6)$$

A specific weighted integral over the scalar spectral function thus directly determines the moments $r_{ij}^{(0)kl}$. (A closely related discussion along these lines, which deals with $J = 1$ and 0 spectral functions and the resulting moments separately and the corresponding predictions can be found in [5].) Since m_u, $m_d \ll m_s$, and $m_s \ll M_\tau$, the function $r_{us}^{k,l}$ is well described by setting $m_u = m_d = 0$ and keeping only the leading and quadratic contributions to the correlator (4), viz.

$$\Pi_{ij}^{(q)}(q^2, m_s) = \frac{3}{16\pi^2}\left(\Pi_0^{(q)}(q^2) + \frac{m_s^2}{Q^2}\Pi_{2,ij}^{(q)}(q^2)\right), \quad (7)$$

$$\Pi_{ij}^{(0)}(q^2, m_s) = \frac{3}{16\pi^2}\frac{m_s^2}{Q^2}\Pi_{2,ij}^{(0)}(q^2). \quad (8)$$

Here $Q^2 = -q^2$. In the following, we limit ourselves by the perturbative contributions (for a recent discussion of power-suppressed contributions, see [30]). The small nonperturbative terms, as well as the m_s^4 terms, are effectively included in the phenomenological analysis at the end. As a result, one has a convenient decomposition

$$r_{ud} = r_0 + \delta_{ud}, \qquad r_{us} = r_0 + \delta_{us}. \quad (9)$$

Let us, in addition, define the difference

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,NS}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} = 3S_{EW}\delta r^{kl}, \quad (10)$$

which is a useful combination to probe the $SU(3)$ breaking effects as $\delta r^{kl} \equiv (\delta_{ud}^{kl} - \delta_{us}^{kl})$ vanishes in the limit of exact $SU(3)$ flavor symmetry. In the spirit of the decomposition (5) one similarly defines related quantities $\delta R_\tau^{(q)kl}$, $\delta R_\tau^{(0)kl}$, $\delta r^{(q)kl}$, and $\delta r_\tau^{(0)kl}$ such that $\delta R_\tau^{kl} = \delta R_\tau^{(q)kl} + \delta R_\tau^{(0)kl}$, $\delta r^{kl} = \delta r^{(q)kl} + \delta r^{(0)kl}$.

The currently known fixed order perturbative predictions for r_0, δ_{us}, and δ_{ud} can be shortly summarized as follows [3,4,31]:

$$r_0 = (1 + a_s + 5.202a_s^2 + 26.37a_s^3), \quad (11)$$

$$\delta_{us} = \frac{m_s^2}{M_\tau^2}(1 + 5.33a_s + 46.0a_s^2), \quad (12)$$

$$\delta_{ud} = -0.35a_s^2\frac{m_s^2}{M_\tau^2}, \quad (13)$$

where $a_s = \alpha_s(M_\tau)/\pi$. Now, with $a_s \approx 0.1$ one observes that the "apparent convergency" of the series is acceptable for r_0 but should be considered at best as marginal for δ_{us}. Clearly, the next term in (12) is important for the interpretation of the measurements. Partial results for the a_s^4 term in Eq. (11) and the a_s^3 term in Eq. (12) have been published [32]. We give the results of the calculation of the missing term as well as its phenomenological implications.

Calculation and results.—To compute the real part of $\Pi_2^{(q)}(q^2)$ we proceed as follows. First, using the criterion of irreducibility of Feynman integrals [16], the set of irreducible integrals involved in the problem was constructed. Second, the coefficients multiplying these integrals were calculated as series in the $1/D \to 0$ expansion. Third, the exact answer, i.e., a rational function of D, was reconstructed from this expansion. Our results read

$$\Pi_{2,us}^{(q)} = -4 - \frac{28}{3}a_s + a_s^2\left\{-\frac{13981}{108} - \frac{646}{27}\zeta_3 + \frac{2080}{27}\zeta_5\right\} + a_s^3\left\{-\frac{2092745}{1296} - \frac{14713}{162}\zeta_3 - 122\zeta_3^2 + 10\zeta_4 + \frac{41065}{27}\zeta_5 - \frac{79835}{162}\zeta_7\right\} = -4(1 + 2.333a_s + 19.58a_s^2 + 202.309a_s^3), \quad (14)$$

$$\Pi_{2,ud}^{(q)} = a_s^2\left\{\frac{128}{9} - \frac{32}{3}\zeta_3\right\} + a_s^3\left\{\frac{6392}{27} - \frac{4496}{27}\zeta_3 - 16\zeta_3^2 + \frac{320}{27}\zeta_5\right\} = -4(-0.35a_s^2 - 6.437a_s^3). \quad (15)$$

Phenomenology.—First of all, it is instructive to compare the exact result for the $O(\alpha_s^3)$ contribution to (14) with the recently obtained predictions [32] based on the optimization schemes as PMS and FAC:

$$k_{2,us}^{(q)3} = 202.309\,(\text{exact}), \quad 201\,(\text{PMS}), \quad 199\,(\text{FAC}). \tag{16}$$

This astonishingly good agreement [a similar phenomenon has been observed for the prediction of the $O(\alpha_s^3)$ term for the correlator of the diagonal currents [33]] can be considered as a strong argument to repeat the procedure and predict, starting from the now completely known $k_2^{(q)3}$, the corresponding result for one loop more, that is, for $k_2^{(q)4}$. To be definite, we use the PMS predictions (again for $n_f = 3$; the FAC result is very similar)

$$k_{2,us}^{(q)4} = 2276 \pm 200 \quad \text{and} \quad k_{2,\text{us}}^{(q)4} - k_{2,\text{ud}}^{(q)4} = 2378 \pm 200. \tag{17}$$

It is, of course, difficult to estimate uncertainty in the above predictions; however, the simple comparison with Eq. (16) clearly demonstrates that an error of about 10% should be considered quite conservative.

Apart from evaluations using fixed order perturbation theory two formally equivalent versions of the contour improved procedure can be found in the literature. The first [5] is based directly on the integration of the polarization function $\Pi_2^{(q)}$, the second [6] is based on the integration of the Adler function $D_2^{(q)} \equiv s\frac{d}{ds}\Pi_2^{(q)}$ and is obtained from the first one by partial integration. Partial integration and renormalization group improvement do not commute as long as finite orders are considered. Since the second procedure moves part of the lower order input to higher orders (contrary to the spirit of CIPT) and since, furthermore, the first procedure leads to a somewhat more stable perturbation series [34], we consider the first of the two choices as preferable. The three options for the perturbative series are displayed in Table I. As a consequence of the large value of α_s and the rapidly growing coefficients of the perturbative series it seems at first glance difficult to consider any of the theoretical predictions as truly preferable. Nevertheless, the results for moments $(2,0)$, $(3,0)$, and $(4,0)$ are at least in plausible agreement among the three methods and exhibit acceptably decreasing subsequent terms. Since these moments are also relatively most precise, as far as experiment is concerned, they are used in the subsequent analysis. The phenomenological analysis is based on the most recent evaluation [35] of $|V_{\text{us}}|$. For the phenomenological description of the contribution due to $r^{(0)kl}$ we adopt the analysis presented in [29] for the second version of contour improvement.

The results for m_s, derived from different moments and different ways of implementing the contour improvement procedure, are shown in Table II. (Details about the error estimates and the corresponding analysis will be given elsewhere.) The main difference, compared to the previous analysis, is a downward shift of m_s by about 5 MeV from the inclusion of the α_s^4 terms and an upward shift by as much as 20 MeV from the new input for $|V_{\text{us}}|$. The ambiguity for the determined value of m_s, including the newly computed α_s^3 term, and the estimate for the α_s^4 term is shown in Table II. The renormalization scale $\mu = \xi M_\tau$ is allowed to vary between $\xi = 1$–1.5. Values of μ lower than M_τ lead to a blowup of α_s and destabilize the result. By "others" we mean all uncertainties (added in quadrature) of the input parameters different from the $O(a_s^3)$ [or $O(a_s^4)$] terms in the perturbative contribution. They include experimental errors in the moments as reported in [2], in $|V_{\text{us}}| = 0.2259(23)$ from [35] as well as uncertainties in the

TABLE I. Contributions of successive orders of in α_s to $\delta r^{(q)kl}$ for the value of $\alpha_s(M_\tau) = 0.334$, and normalized to the value of $\delta r^{(q)kl}$ in the Born approximation. First five lines: fixed order perturbation theory, second five lines: contour improved version, with RG improvement for the Adler function $D^{(q)}$; last five lines: contour improved version with RG summation made for the polarization operator $\Pi^{(q)}$.

(kl)	Perturbative series
$(0,0)$	$1 + 0.425 + 0.283 + 0.178 + 0.0987 = 1.98$
$(1,0)$	$1 + 0.532 + 0.458 + 0.423 + 0.437 = 2.85$
$(2,0)$	$1 + 0.606 + 0.589 + 0.623 + 0.734 = 3.55$
$(3,0)$	$1 + 0.663 + 0.695 + 0.793 + 1.00 = 4.15$
$(4,0)$	$1 + 0.708 + 0.784 + 0.943 + 1.24 = 4.68$
$(0,0)$	$0.753 + 0.214 + 0.065 - 0.0611 - 0.213 = 0.76$
$(1,0)$	$0.912 + 0.334 + 0.192 + 0.0675 - 0.0969 = 1.41$
$(2,0)$	$1.05 + 0.451 + 0.33 + 0.228 + 0.0802 = 2.14$
$(3,0)$	$1.19 + 0.571 + 0.484 + 0.425 + 0.33 = 3.0$
$(4,0)$	$1.32 + 0.697 + 0.657 + 0.665 + 0.664 = 4.01$
$(0,0)$	$0.857 + 0.122 - 0.0090 - 0.158 - 0.355 = 0.458$
$(1,0)$	$1.11 + 0.232 + 0.11 - 0.0417 - 0.280 = 1.13$
$(2,0)$	$1.35 + 0.347 + 0.251 + 0.124 - 0.121 = 1.95$
$(3,0)$	$1.59 + 0.471 + 0.42 + 0.35 + 0.145 = 2.97$
$(4,0)$	$1.83 + 0.61 + 0.623 + 0.648 + 0.544 = 4.26$

TABLE II. Result for m_s, derived from different levels of approximation, based on contour improvement from [32] and a list of different contributions to the associated error.

Parameter	Value	$(2,0)$	$(3,0)$	$(4,0)$
$m_s(O(a_s^3)$, exact)		123.0	103.0	88.0
$O(a_s^3)$	$2\times O(a_s^3)$ no $O(a_s^3)$	-3.5 3.9	-5.8 7.0	-6.8 8.9
ξ	1.5 1	-1.4 0	4.5 0	7.5 0
$\alpha_s(M_\tau)$	0.334 ± 0.022	3.8 -1.4	0.44 1.2	-1.5 2.8
Others [2,29,35]		$+22.3$ -25.7	$+17.0$ -19.1	$+14.3$ -15.8
Total		$+23.0$ -26.0	$+19.0$ -20.0	$+18.6$ -17.3
$m_s(O(a_s^4)$, PMS)		127.0	100.0	82.4
$O(a_s^4)$	$2\times O(a_s^4)$ no $O(a_s^4)$	3.9 -3.6	-2.3 2.4	-4.6 5.6
ξ	1.5 1	-9.1 0	-2.0 0	4.4 0
$\alpha_s(M_\tau)$	0.334 ± 0.022	13.0 -5.6	4.3 -0.71	0.24 2.1
Others [2,29,35]		$+22.9$ -26.4	$+16.7$ -18.8	$+13.8$ -15.3
Total		$+26.7$ -28.6	$+17.5$ -19.0	$+15.7$ -16.0

parameters related to construction of the subtracted longitudinal part. We have computed the uncertainties using numbers from [29].

From Table I we find that the inclusion of the α_s^4 term leads to a better agreement between predictions based on two different methods of implementing the "contour improvement" approach for the third and the fourth moments. As the former moment also shows a smaller theoretical error involved, we choose it to derive our final result for m_s, which is given below.

In total we find

$$m_s(M_\tau) = 100 + \begin{pmatrix} +5.0 \\ -3.0 \end{pmatrix}_{\rm theo} + \begin{pmatrix} +17.0 \\ -19.0 \end{pmatrix}_{\rm rest} \text{MeV}. \quad (18)$$

If one compares (18) to the $O(\alpha_s^3)$ result of [29] (corrected for a different $|V_{\rm us}|$) $m_s(M_\tau) = 106$ MeV (with larger theoretical and identical remaining errors), one sees an essential but still not too large sensitivity to the α_s^4 contribution. In fact, for the third moment the shift in m_s due to inclusion of the α_s^4 term (-2.5 MeV) is about a third of the corresponding change (-7 MeV) due to the α_s^3 contribution. Thus, the purely theoretical uncertainty from not yet computed higher orders could be estimated as about 3 MeV. Unfortunately, one can hardly hope that the error from not yet computed higher orders in α_s could be reduced further by means of a direct calculation in any foreseeable future.

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