Relations between Cloning and the Universal NOT Derived from Conservation Laws

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We discuss certain relations between cloning and the NOT operation that can be derived from conservation laws alone. Those relations link the limitations on cloning and the NOT operation possibly imposed by *other* laws of nature. Our result is quite general and holds both in classical and quantum-mechanical worlds, for both optimal and suboptimal operations, and for bosons as well as fermions.

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As is well known, there are fundamental limitations on the accuracy of certain quantum operations, with cloning and the NOT operation applied to quantum bits being two prime examples [1–9]. Those limitations are independent of physical implementation. For instance, it is irrelevant whether the qubits are implemented using Josephson junctions, ions, or photons. Similarly, it is irrelevant whether the two basis states $|0\rangle$ and $|1\rangle$ of the qubit correspond to eigenstates of different charge, of different energy, or of different angular momentum. Once one has chosen a particular implementation, however, there are often, if not always, conservation laws that must be obeyed. Indeed, there must be at least one physical quantity that takes different values in the states $|0\rangle$ and $|1\rangle$; otherwise the two states would not be orthogonal and distinguishable. Depending on the situation there will be a conservation law for that quantity, or for the complementary variable, or for both. For example, although there are no conservation laws for position or time, there are for linear momentum and energy.

In the following we consider relations between limitations on cloning and the NOT operation that arise from two simple assumptions: (1) The states $|0\rangle$ and $|1\rangle$ correspond to eigenstates of some conserved quantity with eigenvalues -1 and $+1$ in some appropriate unit. For concreteness we will say that the states are eigenstates of "angular momentum." (2) Each qubit is a "particle" and a conservation law holds for the number of particles. In relation to assumption (2), note that in certain contexts (when considering atoms, ions, quantum dots, or any other material entities as qubits) conservation of the particle number is appropriate, whereas in other contexts (for example, when one considers photons as qubits [7–9]) conservation of the excitation number is more appropriate. In the following considerations these two cases are mathematically equivalent, and for ease of notation we will henceforth refer to particles and use the particle conservation law.

It is important to note that in spite of the quantum-mechanical notation and terminology used here, the assumptions just mentioned by themselves make no use of quantum mechanics. In particular, here and in the following we will only need to discuss particles in states $|0\rangle$ and $|1\rangle$, but not in superpositions of $|0\rangle$ and $|1\rangle$. The relations we will find between cloning and NOT operations hold, therefore, just as well for classical cloning procedures and NOT operations. But since in the end we are mostly interested in understanding the quantum-mechanical results, we adopt quantum-mechanical notation.

Suppose we start out with N particles in the state $|0\rangle$ and attempt to generate $M > N$ clones in the same state. In general, we will end up not only with M clones in states $|0\rangle$ and possibly in state $|1\rangle$, but, by assumption (1), with some nonzero number K (to be determined later) of ancilla particles that must be present to compensate for the amount of angular momentum produced or destroyed in the cloning process. By assumption 2 then, we must have borrowed $M + K - N$ particles from elsewhere, a "reservoir" of particles. We assume the reservoir starts out in a state with equal numbers of particles, say L, in states $|0\rangle$ and $|1\rangle$. Thus, we denote the initial state by

$$|N, 0\rangle \otimes |L, L\rangle \tag{1}$$

where $|n, m\rangle$ denotes a state with n particles in state $|0\rangle$ and m particles in state $|1\rangle$. The attempted cloning operation may then be described by the transformation

$$|N, 0\rangle \otimes |L, L\rangle \mapsto \sum_{a,b} A_{a,b}|a, M - a\rangle \otimes |b, K - b\rangle \otimes |L', L'\rangle. \tag{2}$$

The first ket on the right-hand side refers to clones, the second ket to ancillas, and the third ket to the reservoir. The coefficients $A_{a,b}$ determine in some unspecified way the probabilities $p_{a,b}$ to find a clones in state $|0\rangle$ and b ancillas in state $|0\rangle$. In quantum mechanics we would have $p_{a,b} = |A_{a,b}|^2$. Although we wrote down a quantum-mechanical superposition we may just as well regard the superposition as a classical probability distribution over the various possible outcomes of the cloning operation. The number $L' = L + (N - M - K)/2$ is fixed by particle number conservation, and $N - M - K$ must be an even number.

Assumption (1) puts a constraint on the numbers a, b. Angular momentum conservation gives

$$2(a + b) = N + K + M. \tag{3}$$

At this point K, the number of ancillas, is still somewhat arbitrary. Namely, after having fixed the cloning operation we can always *increase* the number of ancillas by taking some (even) number of particles from the reservoir and promoting them to ancillas. This does not affect the cloning operation and, provided we transfer equal numbers of particles in states $|0\rangle$ and $|1\rangle$ from the reservoir, does not affect angular momentum conservation of clones and ancillas either. It makes sense then to take the *smallest* possible number K consistent with all conservation laws as the canonical number of ancillas. That minimum number is easy to determine from constraint (3): suppose nature allows us at least sometimes to achieve perfect cloning of the state $|0\rangle$ with some nonzero probability. In that case we have a term in the superposition with $a = M$. The minimum b allowed is, of course, $b = 0$, so that the minimum K consistent with (3) is

$$K = M - N. \tag{4}$$

Adopting this value for K then fixes b to be

$$b = M - a. \tag{5}$$

With these ingredients, the cloning operation (2) can be rewritten as

$$|N, 0\rangle \otimes |L, L\rangle \mapsto \sum_a A_{a,M-a}|a, M - a\rangle \otimes |M - a, a - N\rangle \otimes |L', L'\rangle. \tag{6}$$

The average cloning fidelity F_{clone} may be defined as

$$F_{\text{clone}} = \sum_a p_a \frac{a}{M}, \tag{7}$$

where p_a (determined by $A_{a,M-a}$) gives the probability to find a clones in the (correct) state $|0\rangle$, and where the cloning fidelity for a state with a clones out of M in the correct state is defined to be the ratio a/M. This in fact corresponds to the standard definition of cloning fidelity, see Ref. [10] and the discussion below.

The ancillas compensate for the angular momentum produced in the cloning procedure and thus, roughly speaking, they will end up in a state with angular momentum opposite to that of the clones. But "flipping the angular momentum" of a qubit state is the same as applying the NOT operation [6]. Thus, the better the cloning procedure works, the better the NOT operation will be implemented on the ancillas. This is why there is a strong connection between the fidelity F_{clone} of the cloning operation and a similar fidelity F_{NOT} one can define for the NOT operation. Namely, F_{NOT} is analogously defined as the average of the ratios of the number of ancillas in state $|1\rangle$, $a - N$, and the total number of ancillas, $M - N$. Thus,

$$F_{\text{NOT}} = \sum_a p_a \frac{a - N}{M - N}. \tag{8}$$

But this immediately gives us a relation between F_{clone} and F_{NOT} that is *independent* of the values of p_a [11]

$$(M - N)F_{\text{NOT}} = MF_{\text{clone}} - N. \tag{9}$$

So far, we only considered cloning and the NOT as applied to $|0\rangle$. But for universal cloners and the universal-NOT operation the fidelities are, by definition, independent of the input state. Thus, the relation (9) holds for any (optimal or suboptimal) universal cloner and universal-NOT operation. The independence of that relation on the details of the transformation (6) demonstrates the generality of the result.

Instead of starting out, as we did here, with an operation that is supposed to clone the state $|0\rangle$, we might as well have begun with describing an operation that is supposed to apply a NOT operation. The constraints we would find then are exactly the same as we found before. And so we would find the same relation (9) again, even if we had found different coefficients $B_{a,b}$ instead of $A_{a,b}$. This, combined with the simple linear relationship between F_{clone} and F_{NOT}, implies, in particular, that optimizing the NOT operation would automatically optimize the cloning operation, and *vice versa*.

The relation (9) quantifies to what extent the NOT operation can be performed given how well cloning can be performed, and *vice versa*. For example, if one can perform one perfectly, the other procedure can be performed perfectly as well. That, of course, reflects what is possible in a classical world, but also what is possible quantum mechanically when one knows the input state. From (9) we see that in general F_{NOT} is never larger than F_{clone}, but in the limit of $M \to \infty$ with N finite one gets $F_{\text{NOT}} = F_{\text{clone}}$. The optimum quantum cloning fidelity for universal cloning of arbitrary unknown input states is well-known to be [3]

$$F_{\text{clone}}^{\text{opt}} = \frac{M(N + 1) + N}{M(N + 2)}. \tag{10}$$

This combined with Eq. (9) immediately yields the optimum universal-NOT fidelity:

$$F_{\text{NOT}}^{\text{opt}} = \frac{N + 1}{N + 2}, \tag{11}$$

which turns out to be independent of M. And indeed, this is identical to the result obtained in [5,6] by other means.

The above-used notation is appropriate for bosons, with a, b being occupation numbers of certain "modes." Nevertheless, the results are equally valid for fermions. Indeed, there are in fact *two* different interpretations of "cloning" when it is applied to bosons, and one of those interpretations applies to fermions as well. For simplicity, first consider $1 \to 2$ cloning. The (arbitrary, unknown) state to be cloned can be written as $A|0\rangle + B|1\rangle$, and cloning, in the standard terminology, would correspond to the transformation

$$A|0\rangle + B|1\rangle \mapsto [A|0\rangle + B|1\rangle]^{\otimes 2}, \tag{12}$$

which adds a second particle and a second system. This formulation works for fermions as well as for bosons. Alternatively, we may write the same state in terms of creation operators $C_{0,1}^\dagger$ for excitations in the different modes 0 and 1. For bosons, but not for fermions, it is possible to create more than one excitation in a single mode. Creating two bosons in the same mode starting from a single boson may also be considered a form of cloning. This would correspond to the transformation

$$[AC_0^\dagger + BC_1^\dagger]|\text{vacuum}\rangle \mapsto \frac{[AC_0^\dagger + BC_1^\dagger]^2}{\sqrt{2}}|\text{vacuum}\rangle, \tag{13}$$

which adds a second particle (excitation) but does not add a second system (mode). Thus, although the initial states in the transformations (12) and (13) are the same, the final states are different. However, it is easy to verify there is a unitary operation taking one final state to the other. In other words, the two descriptions are unitarily equivalent. It is also easy to check that the fidelity in terms of occupation numbers of modes used in the present paper [corresponding to cloning in the sense of (13)] is equivalent to the usual definition of fidelity provided the cloning operation is symmetric [i.e., all clones in Eq. (12) end up in the same state]. This very same point was made in Ref. [10], where details on the equivalence of the two definitions of fidelity can be found. Hence, the limits on $1 \rightarrow 2$ cloning are in fact exactly the same irrespective of which definition of cloning one prefers.

More generally, one can show the two different final states that appear in general $N \rightarrow M$ cloning procedures are unitarily equivalent, and that the two corresponding definitions of fidelity are identical for symmetric cloners. In addition, in a similar manner one may use two different definitions and descriptions for the NOT operation acting on bosons. In the end those two formulations, too, are equivalent, with one of them applicable to fermions.

The *optimal* universal cloner applied to the polarization degree of freedom of photons can be and has been implemented using stimulated emission [7–9]. In that context, the operation (6) can be understood as follows: the initial state consists of N photons in a particular spatial mode (denoted by "1"), all σ^- polarized. The reservoir consists of $2L$ excited atoms: for instance, if one uses a $J = 1/2 \rightarrow J' = 1/2$ transition, then an atom in the excited state $|J_z' = \pm 1/2\rangle$ "stores" a $\sigma^\pm$-polarized photon. By stimulated emission one produces with some probability M photons in the same spatial mode 1 (the clones), and $M - N$ photons in a different spatial mode 2 (the ancillas). The total number of atomic and photonic excitations is conserved, i.e., $2L - 2L' = 2M - 2N$ atoms have decayed to the appropriate ground states. Obviously, angular momentum is conserved too in this case, as expressed by selection rules. The unitary operation implementing the optimal cloner and the optimal NOT corresponding to this particular physical implementation is given in Refs. [7–9].

In conclusion then, we have shown a strong relation exists between (universal) cloning and the (universal) NOT operation. That a relation exists between the *optimum* fidelities for quantum cloning and the quantum universal NOT operations had been noticed before in the context of photons, as we have just mentioned, in Refs. [7–9]. But here we demonstrate that the relation (9) holds more generally: it holds for suboptimal procedures, it holds for fermions as well as bosons, and it holds in the classical world. Moreover, we explain *why* an optimum cloner also implements the optimal NOT operation. The only assumptions needed to derive this are simple conservation laws. Conversely, it had been noted before that for the optimal cloner a conservation law holds: a particularly nice form of such a conservation law can be found in Ref. [12].

Finally, we note that the impossibility of the *perfect* NOT operation arises from it not being a completely positive map [5], whereas no cloning arises from the linearity of quantum mechanics [1]. On the other hand, the optimum fidelities of the corresponding *imperfect* quantum operations are determined by the unitarity of quantum mechanics. Equation (9), however, uses none of those properties: Unitarity or linearity or complete positivity put restrictions on the values of the coefficients $A_{a,b}$ in Eq. (2), but relation (9) holds independent of the precise values of $A_{a,b}$.

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