Anderson Transition in Quantum Chaos

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We investigate the effect of classical singularities on the quantum properties of nonrandom Hamiltonians. A kicked rotator with a nonanalytical potential is discussed in detail. It is shown that classical singularities produce anomalous diffusion in the classical phase space. Quantum mechanically, the eigenstates of the evolution operator are power-law localized with an exponent given by the type of classical singularity. For logarithmic singularities, the classical motion presents $1/f$ noise and the quantum properties resemble those of an Anderson transition. Neither the classical nor the quantum properties depend on the details of the potential but only on the type of singularity. We thus define a new universality class in quantum chaos by the relation between classical singularities (anomalous diffusion) and quantum power-law localization.

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The description of quantum features based on the underlying classical dynamics has been intensively investigated since the early days of quantum mechanics. For instance, the celebrated Bohigas-Giannoni-Schmit conjecture [1] states that the spectral correlations of a quantum system whose classical counterpart is fully chaotic depend only on the global symmetries of the system and are identical to those of a random matrix with the same symmetry [usually referred to as Wigner-Dyson statistics (WD)]. However, this conjecture is not always verified. A paradigmatic example is the kicked rotor: Although classically it is chaotic, their eigenfunctions are exponentially localized. This feature is fully understood [2] after mapping the kicked rotator problem onto a 1D disordered system (free particle in a random potential), where it has been well established that, in less than three dimensions, all the eigenstates are exponentially localized for any amount of disorder. In the context of disordered systems a metal insulator transition (MIT), also called the Anderson transition, can be observed in more than two dimensions for a critical amount of disorder. At the MIT, the eigenfunction moments $\mathcal{P}_q$ present anomalous scaling with respect to the sample size L, $\mathcal{P}_q = \int d^d r |\psi(\mathbf{r})|^{2q} \propto L^{-D_q(q-1)}$, where D_q is a set of exponents describing the transition. Wave functions with such a nontrivial scaling are said to be multifractals [3]. Spectral fluctuations at the MIT are commonly referred to as "critical statistics" [4]. Typical features include scale invariant spectrum, level repulsion, and linear number variance. Similar properties have also been observed in disordered systems with random power-law hopping [5] (linked to classical anomalous diffusion [6]) provided that the exponent of the hopping decay matches the dimension of the space. In the 1D case ($1/r$ decay) a MIT has been analytically established [7]. For *nonrandom* power-law hopping, a MIT was recently reported even for power-law exponents larger than the dimensionality of the space [8]. Unlike the random case, critical states appear only in a certain energy window. Similar properties also have been found in certain nonrandom Hamiltonians: Fermi accelerator [9], Coulomb billiard [10], anisotropic Kepler problem [11], and generalized kicked rotors [12,13]. In all of them the potential has a singularity, and consequently the Kol'mogorov-Arnol'd-Moser theorem (KAM) theorem does not hold. However, their classical phase spaces are fractal resembling the cantori structure typical of the border between chaotic and integrable regions in KAM systems. Indeed, recent results [14] suggest that the quantum effects of cantori are similar to a MIT.

The goal of this Letter is to show that, in certain cases, classical chaotic Hamiltonians with nonanalytical potentials present classical anomalous diffusion and quantum power-law localization of the eigenstates. The case of logarithmic singularities, leading to a MIT, is investigated in detail.

Classical mechanics.—We study the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + V(q)\sum_n \delta(t - nT) \tag{1}$$

with $V(q) = \epsilon|q|^\alpha$ and $V(q) = \epsilon \log(|q|)$; $q \in [-\pi, \pi)$, $\alpha \in [-1, 1]$, where ϵ is a real number. The evolution over a period T is dictated by the map $p_{n+1} = p_n - \frac{\partial V(q_n)}{\partial q_n}$, $q_{n+1} = q_n + T p_{n+1}$ (mod 2π). Our aim is to show that, though obviously deterministic, the motion is accurately modeled by an anomalous diffusion process. In particular, we compute the classical density of probability $P(p, t)$ for finding the particle at time t with a momentum p. Unlike the case of normal diffusion, the information obtained from the knowledge of a few moments of the distribution is not sufficient to fully characterize the classical motion [15]. The classical density of probability $P(p,t)$ is evaluated by evolving 10^8 initial conditions uniformly distributed in $(q,p) \in [-\pi, \pi) \times [-\pi, \pi)$. For $\alpha > 0.5$ (see Fig. 1) the diffusion is normal and $P(p,t) \sim e^{-cp^2/t}/\sqrt{t}$ (c is a constant). However, for $-0.5 < \alpha < 0.5$ and $t > p$, $P(p,t) \sim tp^{\gamma(\alpha)}$ has power-law tails (see inset of Fig. 1 for a relation between γ and α). Such tails are considered a signature of anomalous diffusion. They are, indeed, a solution of the fractional diffusion equation, $(\frac{\partial}{\partial t} - \frac{\epsilon}{2}\frac{\partial^{-\gamma-1}}{\partial |p|^{-\gamma-1}})P(p,t) = \delta(p)\delta(t)$ (see [16] for the definition of fractional derivative). In order to explicitly show that the origin of the anomalous diffusion is exclusively related to the classical singularity, we have repeated the calculation with the singularity-free potential $V(q) = \epsilon(|q| + b)^\alpha$, $b > 0$. As expected, the diffusion goes back to normal as in the kicked rotor with a smooth potential (see Fig. 1). Also we have checked that these results are generic; namely, they do not depend on the details of the potential but only on the kind of singularity. Thus $P(p,t)$ is not altered by adding a smooth perturbation $V_{\rm per}$ in Eq. (1) (see inset of Fig. 2). The potential $V(q) = \epsilon \log|q|$ has been studied in detail since its quantum counterpart (see below) resembles a disordered conductor at the MIT. In this case (see Fig. 2) $P(p,t)$ is accurately fitted by a Lorentzian. We recall this special form of $P(p,t)$ describes the dynamics of systems with $1/f$ noise [16]. Finally, we mention that the classical phase space is fractal resembling the cantori structure typical of the border between chaotic and integrable regions in KAM systems.

Quantum mechanics.—We now discuss the quantum properties of Eq. (1). For the sake of clearness let us first state our main results: (1) The eigenstates of the evolution matrix associated with the Hamiltonian Eq. (1) are power-law localized with an exponent depending only on the singularity α. (2) For $V(q) = \epsilon \log|q|$, the eigenstates are multifractals and the spectral correlations are described by critical statistics as at the MIT. The quantum dynamics of a periodically kicked system is governed by the quantum evolution operator $\mathcal{U}$ over a period T. After a period T, an initial state ψ_0 evolves to $\psi(T) = \mathcal{U}\psi_0 = e^{-i\hat{p}^2T/4\hbar}e^{-iV(\hat{q})/\hbar}e^{-i\hat{p}^2T/4\hbar}\psi_0$ where $\hat{p}$ and $\hat{q}$ stand for the momentum and position operator. We wish to solve the eigenvalue problem $\mathcal{U}\Psi_n = e^{-i\kappa_n/\hbar}\Psi_n$ where Ψ_n is an eigenstate of $\mathcal{U}$ with quasieigenenergies κ_n. In order to proceed we first express the evolution operator in a matrix form $\langle m|\mathcal{U}|n\rangle$ in a basis of momentum eigenstates $\{|n\rangle = \frac{e^{in\theta}}{\sqrt{2\pi}}\}$. In order to keep $\mathcal{U}$ unitary for finite N, we have made the momentum periodic with a large period $P = N$ [17]. The kinetic term of the evolution matrix has also been made periodic by setting $T = 2\pi M/N$ with M an integer (not a divisor of N). M is chosen to make T roughly constant (≈ 0.1 in this Letter) for every N used. Based on the relation between the period in momentum P and the period in position Q ($PQ = 2\pi N\hbar$), $\hbar = 1$. The resulting evolution matrix then reads

$$\langle m|\mathcal{U}|n\rangle = \frac{1}{N}e^{-iTn^2}\sum_l e^{i\phi(l,m,n)}, \quad (2)$$

where $\phi(l,m,n) = 2\pi(l+\theta_0)(m-n)/N - iV(2\pi(l+\theta_0)/N)$, $l = -(N-1)/2, \ldots, (N-1)/2$, and $0 \le \theta_0 \le 1$; θ_0 is a parameter depending on the boundary conditions ($\theta_0 = 0$ for periodic boundary condition). The eigenvalues and eigenvectors of $\mathcal{U}$ are then computed by using standard diagonalization techniques. For $\theta_0 = 0$, parity is a good quantum number, and consequently states with different parities must be treated separately. We first study the spectral fluctuations of the unfolded spectrum. The analysis of short range spectral correlations as the level spacing distribution $P(s)$ indicates (see Fig. 3) that for $\alpha > 1/2$ ($\alpha < -1/2$), $P(s)$ tends to Poisson (WD) as the volume is increased. In the intermediate region $-1/2 < \alpha < 1/2$ (not shown), poor statistics and finite size effects make it

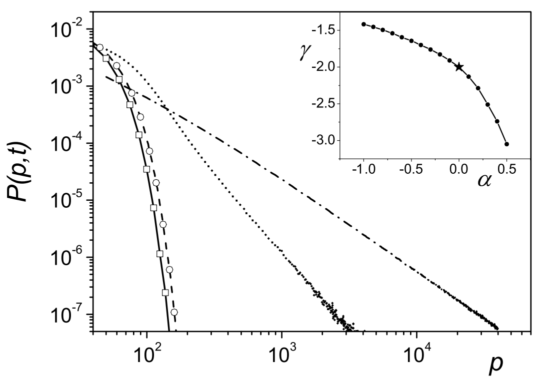

FIG. 1. $P(p,t)$ versus p (log scale). In all cases $t = 2000$ except for $\alpha = -0.5$ where $t = 40$. For $V(q) = |q|^\alpha$ with $\alpha = 0.9$ (solid line) and $V(q) = (|q| + 0.005)^{0.4}$ (dashed line) the diffusion is normal (squares and circles are for best Gaussian fittings, respectively). However, for $\alpha = 0.4$ (dotted line) and $\alpha = -0.5$ (dash-dotted line), $P(p,t) \propto tp^{\gamma(\alpha)}$. Inset shows γ as a function of α. The star corresponds to $V(q) = \log|q|$ (see Fig. 2).

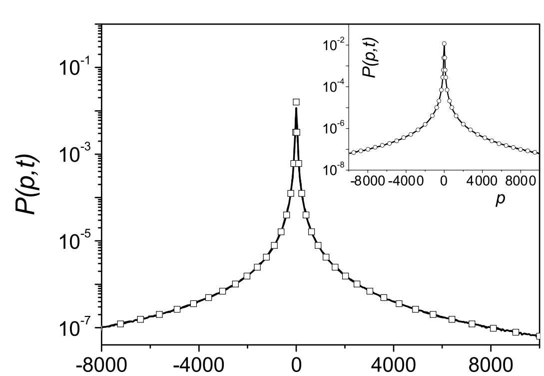

FIG. 2. $P(p, t = 40)$ versus p (semilog scale). For $V(q) = \log|q|$ (solid line) $P(p,t)$ is very well fitted by the Lorentzian distribution $P(p,t) = \frac{1}{\pi}\frac{\nu(t)}{1+p^2\nu^2(t)}$ with $\nu(t) = 2/t$ (squares). As shown in the inset, $P(p,t)$ (solid line) is not affected by adding a noise term $\cos(q)$ to $V(q)$ (circles).

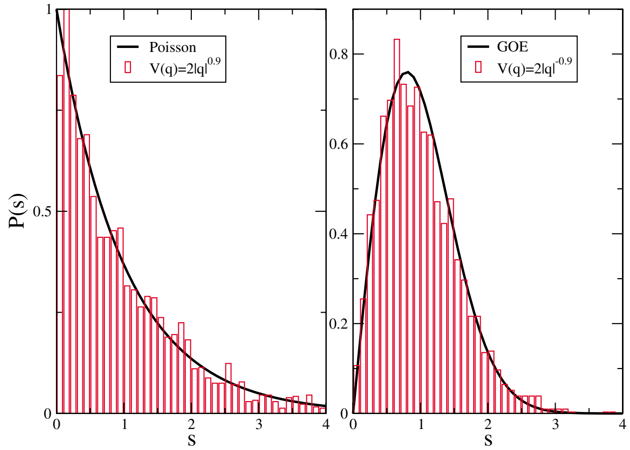

FIG. 3 (color online). The level spacing distribution $P(s)$ (for $N = 3100$) in units of the mean level spacing $s = (E_i - E_{i-1})/\Delta$. A transition is observed from Poisson to WD statistics (keeping ϵ constant) as α goes from negative to positive.

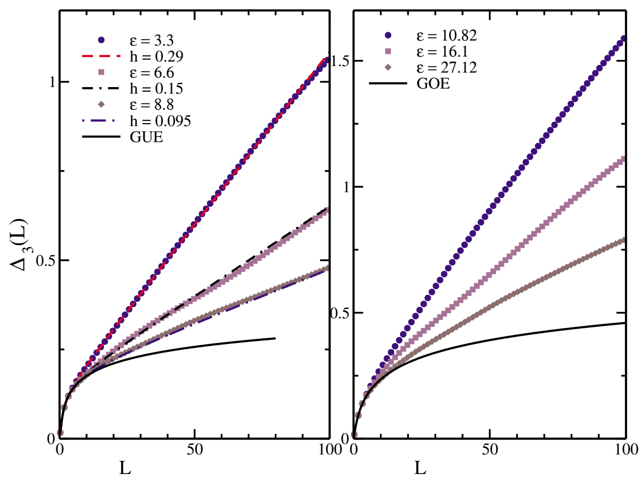

FIG. 4 (color online). The spectral rigidity $\Delta_3(L)$ versus L. Symbols are the numerical calculations for $V(q) = \epsilon \log|q|$. Lines are the analytical prediction Eq. (3). The right (left) panel corresponds to the case of (broken) time reversal invariance.

hard to get a precise evaluation of $P(s)$. However, for $\alpha \sim 0$, we have observed typical features of a MIT as $P(s) \sim s$ for $s \ll 1$ but exponential decay for $s \gg 1$. Long-range correlators, such as $\Delta_3(L) = \frac{2}{L^4}\int_0^L (L^3 - 2L^2 x + x^3)\Sigma^2(x)dx$ where $\Sigma^2(L) = \langle L^2\rangle - \langle L\rangle^2 = L + 2\int_0^L (L - s)R_2(s)ds$ is the number variance and $R_2(s)$ is the two level spectral correlations function, are more suitable for a qualitative analysis due to the possibility of performing a spectral average. We restrict ourselves to the potential $V(q) = \epsilon \log|q|$ related to classical $1/f$ noise. The $R_2(s)$ associated with this type of motion has been evaluated in Ref. [6] for systems with broken time reversal invariance:

$$R_2(s) = -K^2(s) = -\frac{\pi^2 h^2}{4}\frac{\sin^2(\pi s)}{\sinh^2(\pi^2 h s/2)}, \qquad (3)$$

where $h \ll 1$ is related to ϵ by $h = 1/\epsilon \ll 1$. In Fig. 4 we show $\Delta_3(L)$ for different ϵ's and $N = 3100$. The time reversal invariance in Eq. (2) is broken by setting $T = 2\pi\beta$ with β an irrational number. As observed in Fig. 4 (left), $\Delta_3(L)$ is asymptotically linear as at the MIT. Moreover, the agreement with the analytical prediction based on Eq. (3) is excellent. We note that the value of h best fitting the numerical result is within 5% of the analytical estimate $h = 1/\epsilon$. We have repeated the calculation keeping the time reversal invariance ($T = 2\pi M/N$) in Eq. (2) (Fig. 4, right). We do not have in this case an analytical result to compare with, but $\Delta_3(L)$ is also asymptotically linear as in the previous case.

The eigenstates $\Psi_n = \sum \psi_n(k)|k\rangle$ of the evolution operator $\mathcal{U}$ are also strongly affected by the classical singularity α. Instead of the exponential localization typical of the standard kicked rotor, our results (see Fig. 5) indicate that, in agreement with [5,7], $|\psi_n(k)| \sim |k - k_{\max}|^{-(\alpha+1)}$ decays as a power law ($k_{\max}$ is the maximum of the eigenfunction) with the exponent controlled by the classical singularity. Thus, both classical (anomalous) dynamics and quantum (power-law) localization are controlled by the classical singularity α. This fact allows us to define a new universality class in quantum chaos labeled by α. A remark is in order: in the critical case $\alpha = 0$ the eigenstates (see below) have a complicated multifractal structure with peaks at all scales which decay as a power law with exponents depending also on ϵ. Consequently, the above single power-law decay is strictly valid only in the $\epsilon \to 0$ corresponding to the situation in which only one of such peaks is present.

We now investigate how transport properties are affected by the nonanalytical potential. We compute the quantum density of probability $P_q(k, t) = |\langle k|\phi(t)\rangle|^2$ of finding a particle with momentum $k\hbar$ at time t for a given initial state $|\phi(0)\rangle = |0\rangle$. We restrict ourselves to the case $V(q) = \epsilon \log|q|$ in order to compare our findings with the ones at the Anderson transition where it is reported [18] that (in

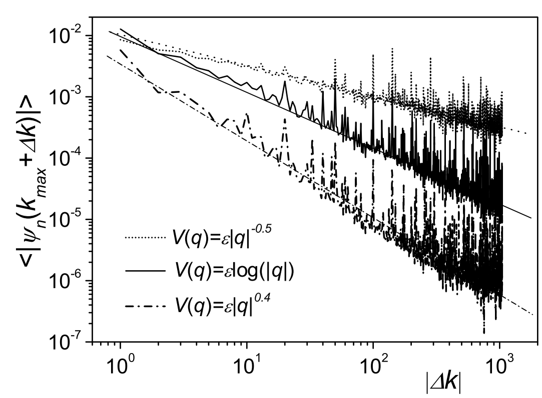

FIG. 5. Averaged modulus of the eigenstates of the evolution operator for $N = 2100$. We set $\epsilon = 0.01$ to a small value in order to avoid finite size effects. The best fitting slopes -0.49, -0.98, and -1.36 corresponding to $\alpha = -0.5, 0(\log), 0.4$, respectively, are close to $-(1 + \alpha)$ [7].

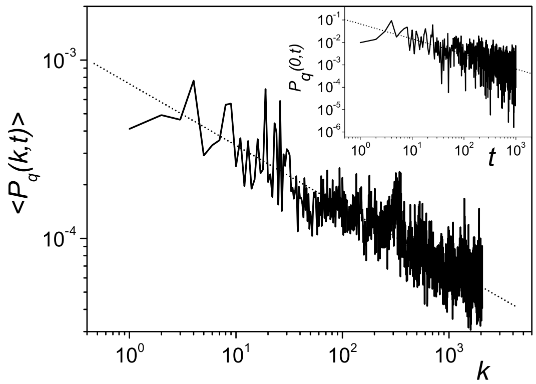

FIG. 6. Time averaged $P_q(k, t)$ over $4950 \le t \le 5050$ and $P_q(k = 0, t)$ versus t (inset) for $V(q) = 10 \log|q|$. The best fitting (dotted line) with slopes -0.34 and -0.67 (inset) are consistent with $D_2 \approx 0.65$.

real space), in the limit $r \ll t/\hbar\rho$ with ρ the density of states, $P_q(r, t) \sim t^{-D_2} r^{D_2 - 1}$ with D_2 the multifractal dimension previously defined. As expected, we have observed a similar power-law behavior in momentum space in our model (see Fig. 6). The numerical value of D_2 increases as ϵ does. This is within 10% of the value of D_2 obtained by scaling $\mathcal{P}_q = \int d^d r |\psi(\mathbf{r})|^{2q} \propto L^{-D_q(q-1)}$. We have also checked that in the $\epsilon \gg 1$ limit $D_q(q-1) \propto q$ as for the random banded model of Ref. [7]. These results show that quantum mechanically the diffusion is also anomalous but, unlike the classical case, the decay of $P_q(k, t)$ depends not only on the classical singularity α but also on the coupling constant ϵ. The overall effect of the quantum corrections is also to suppress the classical (anomalous) diffusion at a rate that increases as ϵ gets smaller. To summarize, quantum $1 + 1$D systems with classical log singularities share all the properties of a disordered conductor at the MIT: Multifractality, critical statistics, and quantum anomalous diffusion. Analytical results can, in principle, be obtained by mapping Eq. (1) onto a 1D Anderson model. This method is introduced in [2] for the case of a kicked rotor with a smooth potential. We do not repeat here the details of the calculation but just state how the 1D Anderson model is modified by the nonanalytical potential. It turns out that classical nonanalyticity induces long-range disorder in the associated 1D Anderson model $T_m u_m + \sum_r U_r u_{m+r} = E u_m$, where $U_r \sim A_r / r^{1+\alpha}$ is the Fourier coefficient of $-\tan V(q)/2$, $T_m = \theta_0/2 - T m^2/2$ and $E = -U_0$. For random Gaussian A_r this model corresponds to the one studied in [7] which is solved by using the supersymmetry method. For the special case of a constant nonrandom A_r (closer to our case) it is found that [8] some of the eigenvectors remain critical even for $1 < \alpha$. This suggests that the results reported above for $\alpha = 0$ (log potential) may be extended to other α's though more work is needed to clarify this issue further.

In conclusion, we have shown that classical singularities in chaotic Hamiltonians may lead to classical anomalous diffusion and quantum power-law localization. Both quantum and classical features are governed by the classical singularity. We thus put forward a new universality class in quantum chaos labeled by the type of classical singularity. In the case of logarithmic singularities the classical dynamics presents $1/f$ noise. Quantum mechanically the system possesses all the features of a MIT: multifractal wave functions, critical statistics, and quantum anomalous diffusion.

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