

Exact Counting of Supersymmetric Black Hole Microstates

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It is shown that the entropy of certain two-charge supersymmetric black holes in $N = 4$ string theories can be computed to all orders using Wald's formula and the supersymmetric attractor equations with an effective action that includes the relevant higher curvature terms. Classically, these black holes have zero area but the attractor equations are still applicable at the quantum level and result in finite quantum area. The quantum corrected macroscopic entropy agrees precisely with the microscopic counting for an infinite tower of fundamental string states to all orders in an asymptotic expansion.

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The entropy of a black hole, to leading order, equals one quarter of the area of its event horizon in units of the Planck length. Thermodynamically, this entropy behaves in every respect like ordinary entropy and hence must have a statistical interpretation in terms of underlying microstates. Recent progress in string theory has shown that this is indeed true for a large class of supersymmetric black holes [1].

The black hole entropy thus provides a valuable clue about the microscopic structure of the theory. In a complete quantum theory of gravity, the above formula is expected to receive subleading corrections. It is important to know if these subleading corrections can also be computed precisely and compared with microscopic counting. In this Letter we show that this is indeed possible for certain supersymmetric black holes that correspond to perturbative supersymmetric states in toroidally compactified superstring theories first noticed in [2,3]. We find that the macroscopic entropy agrees with microscopic counting not only to leading order but to *all orders* in an asymptotic expansion in inverse area.

Consider heterotic string compactified on $\mathbf{T}^4 \times \mathbf{T}^2$ where \mathbf{T}^4 is a 4-torus in {6789} directions and \mathbf{T}^2 is a 2-torus which we take to be a product of two circles in the {45} directions. Consider now a string state with winding number w along the x_5 direction. In a given winding sector, there is a tower of half-supersymmetric BPS states each in the right-moving ground state but carrying arbitrary left-moving oscillations subject to the Virasoro constraint $N_L = 1 - nw$, where N_L is the left-moving oscillation number and n is the quantized momentum along x_5 [2,3]. Note that N_L is positive and hence a BPS state that satisfies this constraint has negative n for positive w for large N_L . We will henceforth denote these states by (n, w) .

The number of such states is summarized conveniently by a partition function

$$Z(\beta) = 16 \sum d_N e^{-\beta N}, \quad (1)$$

where $N \equiv w|n| = N_L - 1$. The factor of 16 comes from the degeneracy of the right-moving supersymmetric ground state. Since N_L is the number operator for the 24 left-moving transverse bosons, the partition function

can be readily evaluated

$$Z(\beta) = \frac{16}{\Delta(q)}, \quad (2)$$

where $\Delta(q)$ is the Jacobi discriminant function with argument $q = \exp(-\beta)$. In terms of the Dedekind eta function $\eta(q)$, the discriminant is given by $\Delta(q) = \eta(q)^{24}$.

The number of states at level N is then given by the inverse Laplace transform

$$d_N = \frac{1}{2\pi i} \int_C d\beta e^{\beta N} \frac{1}{\Delta(e^{-\beta})}. \quad (3)$$

To find the asymptotic density at large N , we want to take the high temperature limit or $\beta \rightarrow 0$. It is convenient to use the modular property of the discriminant

$$\Delta(e^{-\beta}) = \left(\frac{\beta}{2\pi}\right)^{-12} \Delta(e^{-4\pi^2/\beta}). \quad (4)$$

As $e^{-4\pi^2/\beta} \rightarrow 0$, we can then use the asymptotics $\Delta(q) \sim q$ and evaluate the integral

$$d_N = \frac{1}{2\pi i} \int_C d\beta e^{\beta N} \left(\frac{\beta}{2\pi}\right)^{12} \frac{1}{\Delta(e^{-4\pi^2/\beta})} \quad (5)$$

in saddle-point approximation. The saddle point occurs at $\beta = 2\pi/\sqrt{N}$ and the degeneracy has the characteristic exponential growth $d_N \sim \exp(4\pi\sqrt{w|n|})$. The subleading terms can be computed in an asymptotic expansion.

This tower of states has played a crucial role in furthering our understanding of dualities and black hole physics. Heterotic string on $\mathbf{T}^4 \times \mathbf{T}^2$ is dual to type IIA on $\mathbf{K}_3 \times \mathbf{T}^2$ [4,5]. Initial evidence for this duality came from matching the low-lying BPS states and the supergravity action but a far more stringent test is obtained by matching the entire infinite tower of BPS states. The state (n, w) is dual to w NS5-branes wrapping $\mathbf{K}_3 \times \mathbf{S}^1$ carrying n units of momentum which in turn is dual to w D4-branes wrapping the \mathbf{K}_3 with a gas n D0-branes on its world volume. The characteristic exponential growth of such brane states was computed in [6]. In fact, this partition function makes its appearance also in topologically twisted Yang Mills theories on \mathbf{K}_3

which provided one of the early hints of the stringy duality [7]. We will return to this dual description subsequently.

Another important application of this tower of states comes from its relation to black hole entropy. The state (n, w) corresponds to a charged extremal black hole in four dimensions [8,9] and one would expect that the logarithm of d_N for large N should match the Bekenstein-Hawking entropy of this black hole. Classically, these black holes have zero area and would appear to have zero entropy, but the higher curvature corrections to the supergravity action can correct the solution. Assuming that the string corrections result in a finite area horizon, it was shown in [10] that the black hole then has nonzero entropy in agreement with the logarithm of the degeneracy. In particular, the nontrivial functional dependence $\sqrt{n|w|}$ on the charges is correctly reproduced. However, the precise numerical factor of 4π could not be computed and the assumption of finite area was in need of further evidence.

In this Letter we show that it is possible to take into account exactly higher curvature corrections to the entropy. After incorporating these corrections and using the exact entropy formula due to Wald [11–13] for the fully corrected action, the precise numerical factor 4π in (5) can be computed. One can even go further and in fact reproduce not only the leading exponential but the entire asymptotic expansion of the partition function (2) exactly to all orders for large N .

Such a detailed computation on the macroscopic side is made possible by the supersymmetric attractor mechanism [14–16], its elegant implementation in supergravity using Wald’s formula for higher derivative F -term corrections pioneered in [17–21], and the recent proposal for the black hole partition function in [22].

Let us summarize the relevant formalism [17–20,22]. To be closer to the discussion in the literature, we work in the dual description of type IIA on $\mathbf{K}_3 \times \mathbf{T}^2$ which can be viewed as a special case of a Calabi-Yau threefold. The resulting supergravity in four dimensions has $N = 4$ supersymmetry but it will be convenient for our purposes to use the $N = 2$ notation of special geometry.

The vector multiplet moduli space of $N = 2$ supergravity with n_v vector multiplets is parametrized by $n_v + 1$ complex projective coordinates $X^I, I = 0, 1, \dots, n_v$. There are an infinite number of higher derivative corrections to the Einstein-Hilbert action that are expected to be relevant for the computation of the entropy. These F -type corrections to the effective action are summarized by the string-loop corrected holomorphic prepotential

$$F(X^I, W^2) = \sum_{h=0}^{\infty} F_h(X^I) W^{2h}, \quad (6)$$

where F_h are computed by the topological string amplitudes [23–27] and W^2 is the reduced chiral multiplet [18,28] that involves the graviphoton field strength. The moduli couple to the electromagnetic fields and, as a result, vary with the radius in the black hole background. Starting

with arbitrary values at infinity, at the horizon they approach attractor values determined by the attractor equations

$$p^I = \text{Re}[CX^I], \quad (7)$$

$$q_I = \text{Re}\left[CF_I\left(X^I, \frac{256}{C^2}\right)\right], \quad (8)$$

and $C^2W^2 = 256$, where $F_I \equiv \partial F/\partial X^I$ are the holomorphic periods. The scaling field C is introduced so that (CX^I, CF_I) is nonprojective and transforms like (p^I, q_I) as a vector under the $Sp(2n_v + 2; \mathbf{Z})$ symplectic duality group. For a recent review of the leading order attractor equations and their applications see [29].

The quantum corrected black hole entropy is given by [17–20]

$$S_{\text{BH}} = \frac{\pi i}{2}(q_I \bar{C}\bar{X}^I - p^I \bar{C}\bar{F}_I) + \frac{\pi}{2}\text{Im}[C^3\partial_C F]. \quad (9)$$

The first set of attractor Eqs. (7) can be solved by

$$CX^I = p^I + \frac{i}{\pi}\phi^I \quad (10)$$

in terms of the “potentials” ϕ^I . Then the entropy (9) can be written in a suggestive form [22] as

$$S_{\text{BH}}(q, p) = \mathcal{F}(\phi, p) - \phi^I \frac{\partial}{\partial \phi^I} \mathcal{F}(\phi, p), \quad (11)$$

in terms of a “free energy” function

$$\mathcal{F}(\phi, p) = -\pi\text{Im}\left[F\left(p^I + \frac{i}{\pi}\phi^I, 256\right)\right]. \quad (12)$$

The potentials ϕ^I in this equation are determined in terms of the charges by the second set of attractor Eqs. (8)

$$q_I = \frac{1}{2}(CF_I + \bar{C}\bar{F}_I) = -\frac{\partial}{\partial \phi^I} \mathcal{F}(\phi, p). \quad (13)$$

Given the form of the entropy (12), it is natural to define a “partition function” as suggested in [22]

$$Z_{\text{BH}}(\phi^I, p^I) = e^{\mathcal{F}(\phi^I, p^I)} \equiv \sum_{q_I} \Omega(q_I, p^I) e^{-\phi^I q_I}, \quad (14)$$

where $\Omega(q_I, p^I)$ are the black hole degeneracies. The Boltzmann entropy $\ln\Omega(q, p)$ is then expected to agree with the thermodynamic entropy $S_{\text{BH}}(q, p)$ in (11) for large charges.

It would be very interesting to test the proposal (14) for the black hole partition function by comparing it with the microscopic partition function. For a general Calabi-Yau compactification, such an explicit comparison is difficult for a number of reasons. On the supergravity side, to make this comparison it is necessary to compute all infinite terms F_h in the prepotential (6). Even though these are given in principle by the topological string, they are not always explicitly computable. On the microscopic side, the counting of states is complicated by the fact that the number of BPS states can jump in $N = 2$ supersymmetric theories

[30]. This phenomenon is possibly related to black hole fragmentation [31] and multiple basins of attraction [32,33] as suggested in [22]. In addition, there are subtleties having to do with the holomorphic anomalies and the background dependence on hypermultiplet moduli which complicate the picture further [22].

One virtue of the tower of states in $N = 4$ compactification that we have considered is that it provides a particularly simple but nontrivial example for making a clean comparison between black hole microstates and the exact entropy formula. For this system, on the macroscopic side the prepotential is explicitly computable. Moreover on the microscopic side the exact partition function of the microstates is also known and is given by (1). With $N = 4$ supersymmetry we do not expect that the number of BPS states would jump.

For type IIA on a Calabi-Yau manifold, in the large volume limit, F_0 and F_1 are given by

$$F_0 = -\frac{1}{6} C_{IJK} \frac{X^I X^J X^K}{X^0}, \quad F_1 = -\frac{1}{64} \frac{c_2}{24} \frac{X^1}{X^0}, \quad (15)$$

where $A = 1, \dots, n_v$ and c_2 is the second Chern class and C_{IJK} are the intersection numbers of a basis $\{\Sigma^I\}$ of 4-cycles [28]. For a properly normalized basis of 2-forms $\{\omega_I\}$ that are Poincaré dual to $\{\Sigma^I\}$, the intersection numbers are given by

$$C_{IJK} = \int_{CY3} \omega_I \wedge \omega_J \wedge \omega_K. \quad (16)$$

In the special case of $\mathbf{K}_3 \times \mathbf{T}^2$, there are 23 2-cycles of which we take w_1 to be the 2-torus itself and w_a , $a = 2, \dots, 23$ to be the 22 2-cycles of \mathbf{K}_3 . The $N = 2$ reduction of $N = 4$ is a bit subtle in supergravity because of the extra gravitini multiplets. For the particular charge configuration that we have chosen, however, the fields in the gravitini multiplets are not excited. Hence we can safely ignore them.

A major simplification for $\mathbf{K}_3 \times \mathbf{T}^2$ is that in (6) all F_h for $h > 1$ vanish. This can be seen most easily in the corresponding topological string from the counting of fermion zero modes. Moreover, F_0 is given by its classical value and receives no quantum corrections because F_0 determines the metric on the moduli space which is known to receive no corrections in $N = 4$ supergravity. Thus, the only nontrivial term in the prepotential comes from F_1 which has already been computed in the literature in a number of different ways—either directly from its definition [24,34], or by using the holomorphic anomaly [23], or from string-string duality [35] by requiring agreement with the Riemann-squared coupling in tree-level heterotic action. The fully quantum corrected prepotential then takes a particularly simple form

$$F(X, W^2) = -\frac{1}{2} C_{ab} X^a X^b \left(\frac{X^1}{X^0} \right) - \frac{W^2}{128\pi i} \log \Delta(q) \quad (17)$$

where $q = \exp(2\pi i X^1/X^0)$ and C_{ab} is the intersection

matrix of $K3$ and we have used the fact that $c_2 = 24$ for \mathbf{K}_3 . It can be seen using the action in [17,36] that X^1/X^0 is the correctly normalized dilaton-axion field λ in the heterotic description that couples to the integral Pontryagin class so that the action is then invariant under $\lambda \rightarrow \lambda + 1$.

Now that we have the exact prepotential, let us see what our charge configuration looks like in this basis. The perturbative state (n, w) on the heterotic side is dual to w 4-branes wrapping the \mathbf{K}_3 with n 0-branes sprinkled on it. The 4-cycle is dual to the 2-form w_1 and hence we have a nonzero magnetic charge $p_1 = w$ and all other magnetic charges are zero. The 0-brane couples electrically to the graviphoton field and hence $q_0 = n$ and all other electric charges are zero. We can readily evaluate the free energy defined in (12),

$$\mathcal{F}(\phi, p) = -\frac{1}{2} C_{ab} \phi^a \phi^b \frac{p^1}{\phi^0} - \log(|\Delta(q)|^2) \quad (18)$$

with

$$q = \exp\left(\frac{2\pi^2 p^1}{\phi^0} + \frac{2\pi i \phi^1}{\phi^0}\right). \quad (19)$$

In the large volume limit we can approximate the second term in (18) by $-4\pi^2 p^1/\phi^0$. It is then easy to solve the attractor equations for the set of charges $q^A = (q^0, 0, 0, \dots, 0)$ and with $p_A = (0, p_1, 0, \dots, 0)$. The solution is

$$\phi^0 = -2\pi \sqrt{\frac{p^1}{|q_0|}}, \quad \phi^a = 0, \quad (20)$$

and ϕ^1 is undetermined. The entropy is given by

$$S = 4\pi \sqrt{p^1 |q_0|} = 4\pi \sqrt{w|n|}, \quad (21)$$

which matches exactly with the logarithm of the degeneracy (5) of the tower of states at large N . It is remarkable that once the higher derivative corrections are included, the attractor formalism is powerful enough to correctly reproduce the entropy even for black holes that have zero area classically. This result is implicit in some of the early work [28], however, there the focus is on computing the corrections to the entropy of black holes that have finite area already classically.

Encouraged by this, we would now like to reproduce the full degeneracy $\Omega(q^A, p_A)$ in (14) which is given by the inverse Laplace transform,

$$\left(\frac{1}{2\pi}\right)^{24} \int \prod_{a=1}^{22} d\phi^a d\phi^1 d\phi^0 \exp[\mathcal{F}(\phi^A, p^A) + \phi^0 q_0]. \quad (22)$$

In the conjectured relation (14), the choice of contours of integrations in (22) and the range of integrations are not clearly specified. Moreover, C_{ab} appearing in (18) has indefinite signature. For an exact evaluation of the integrals in (22), these subtle issues have to be carefully addressed. We instead proceed to evaluate the integrals in a saddle-

point approximation for large charges using contours of steepest descent. Using the free energy (18), the Gaussian integrals over $\{\phi^a\}$ can then be performed to obtain

$$\Omega(q_0, p^1) \sim \frac{(p^1)^2}{2\pi} \int dx \left(\frac{x}{2\pi}\right)^{12} e^{Nx} \int \frac{d\theta}{2\pi} \frac{1}{|\Delta(q)|^2}, \quad (23)$$

where we have defined $x = -\phi^0/p^1$ and $\theta = \phi^1/\phi^0$ so that $q = \exp(2\pi i\tau)$ with $\tau = \theta + \pi i/x$ and $N = -p^1 q_0 \equiv w|n|$. For large N , since the saddle point is localized in the small x region, we can use the approximation

$$|\Delta(q)|^2 \sim |q|^2 \sim e^{-4\pi^2/x} \quad (24)$$

in (23). The θ integral then gives an overall constant which we choose to be $1/(p^1)^2$ to cancel the unwanted, duality noninvariant factor of $(p^1)^2$ in the numerator of (23). We hope that the conceptual reason for this choice of the range of integration will be clarified in future once the formula (14) is better understood. The x integral is then identical to the asymptotic form of the β integral (5) in the region of small β where the saddle point is localized. Hence, up to a multiplicative normalization, the black hole degeneracy of states (23) matches with the microscopic degeneracy of states (5) exactly to all orders in an asymptotic expansion at large N . A more complete analysis will be presented elsewhere [37].

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