Stationary Vortical Flows in Two-Dimensional Plasma and in Planetary Atmospheres

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We derive the equation governing the asymptotic stationary states generated by decaying turbulence in two-dimensional plasma and planetary atmosphere. These fluids may be described by the Charney-Hasegawa-Mima equation and their relaxation states show a high degree of organization in vortical flows, similar to the Euler fluid. We develop a field-theoretical framework and show that these systems attain at stationarity the extremum of an energy functional corresponding to self-dual fields.

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The drift wave in plasma placed in strong magnetic field and the nonlinear Rossby wave in the planetary atmosphere can be described in a convenient approximation by a single scalar field in two-dimensional geometry. This scalar ϕ is the stream function for the Rossby wave and the electrostatic potential for the plasma drift wave, and it obeys the nonlinear differential equation

$$(1-\nabla_\perp^2)\frac{\partial\phi}{\partial t} - v_*\frac{\partial\phi}{\partial y} - [(-\nabla_\perp\phi\times\hat{\mathbf{n}})\cdot\nabla_\perp]\nabla_\perp^2\phi = 0.$$

This equation has been derived by Charney in the physics of atmosphere [1] and by Hasegawa and Mima in plasma physics [2]. In the following we will only use plasma physics terms and the equivalence can be found in Ref. [3]. Then ϕ is the electrostatic potential, $\hat{\mathbf{n}}$ is the direction of the magnetic field, (x, y) are the radial and poloidal coordinates in a plasma confinement geometry (like in tokamak), and $v_*\hat{\mathbf{e}}_y = \hat{\mathbf{n}}\times\nabla_\perp \ln n_0$, with $n_0(x)$ the density and $\hat{\mathbf{e}}_y$ the versor along y. The physical (superscript phys) variables are normalized as: $\phi = |e|\phi^{\text{phys}}/T_e$, $(x, y) = (x^{\text{phys}}/\rho_s, y^{\text{phys}}/\rho_s)$, $t = t^{\text{phys}}\Omega_{ci}$, where $\Omega_{ci} = |e|B/m_i$, $\rho_s = c_s/\Omega_{ci}$, $c_s^2 = T_e/m_i$. Here B is the magnetic field, $|e|$ and m_i are the ion electric charge and mass, and T_e is the electron temperature.

Numerical studies of the Charney-Hasegawa-Mima (CHM) equation [4–6] have shown that in the absence of drive and with very low viscosity the plasma evolves to states of organized flow consisting of few vortices with regular shape. This is very similar to the case of the two-dimensional ideal Euler fluid, where it has been found that the relaxation states are highly ordered and the stream function obeys the sinh-Poisson equation [7,8] (and references therein). The sinh-Poisson equation has first been inferred from numerical experiments and later derived along two lines of argumentation. In the first one it has been considered a statistical system of pointlike vortices interacting in plane by a potential expressed as the natural logarithm of the relative distance [9,10]. The equation corresponds to the most probable state. The second approach has developed a field-theoretical formalism for the continuum limit of the discrete vortices model and has shown that its action functional is extremized by self-dual states (which saturate the Bogomol'nyi bound) [11]. The fluid evolves to states that in field theory are known as self-dual.

There is a discrete vortex model for the CHM equation as well. It has been proposed by Stewart [12] and Morikawa [13] and consists of a discrete set of N pointlike vortices with vorticity ω_k with the equations of motion

$$-2\pi\omega_k\frac{dx_k}{dt} = \frac{\partial W}{\partial y_k} \qquad -2\pi\omega_k\frac{dy_k}{dt} = -\frac{\partial W}{\partial x_k}, \tag{1}$$

where $W = \pi\Sigma_{j\neq i}^N\omega_i\omega_j K_0(|\mathbf{r}_i - \mathbf{r}_j|)$ is the Kirchhoff function and K_0 is the modified Bessel function. We insert explicitly the physical space scale, $m = \rho_s^{-1}$, and define the partial contributions $\psi_j(\mathbf{r})$ to the potential ψ

$$\psi(\mathbf{r}) = \sum_{j=1}^N \psi_j(\mathbf{r}) \equiv \sum_{j=1}^N \omega_j K_0(m|\mathbf{r}-\mathbf{r}_j|). \tag{2}$$

We note that they verify the equation $(\Delta - m^2)\psi_j(\mathbf{r}) = -2\pi\omega_j\delta(\mathbf{r}-\mathbf{r}_j)$ and that the equations of motion (1) can be written $d\mathbf{r}/dt = -\nabla\psi\times\hat{\mathbf{n}}$. This has the same form as in the Euler fluid case, but there the potential ψ is the Green function of the Laplace equation, i.e., $m = 0$. In the CHM case, the pointlike vortices interact via a potential with short range. This is the intrinsic spatial scale of the CHM equation $\rho_s = m^{-1}$ [3].

The original models for Euler and CHM fluids are expressed in terms of stream function, velocity, and vorticity, fields that have clear physical meaning. Relative to this, the fundamental characteristic of the equivalent discrete vortex models is that they exhibit a different structure: matter (vortices), field (the potential), and interaction. Because of this characteristic the continuum versions of these models can be formalized as field theories. We will do this for the CHM discrete vortex model by constructing a Lagrangian density. It will result that the action functional is extremized by a particular subset of states, corresponding to self-duality. As in the case of Euler fluid, the CHM fluid evolves to these states. We find the differential equation that governs these states.

The Lagrangian density should contain (1) a term for the free gauge field that produces the potential of interaction between the vortices; (2) terms for the matter (related to the density of vortices); there will be terms for the kinematic part and for the nonlinear self-interaction; (3) minimal coupling between gauge and scalar fields, via the covariant derivatives.

The potential in the CHM model has short range (finite-mass "photon") and can be written as a derivative of the angle between $\mathbf{r} - \mathbf{r}_j$ and an arbitrary direction. This results from the relation $\varepsilon^{\alpha\beta} r^\beta / r^2 = -\partial\theta/\partial r^\alpha$ and noticing that in the equations of motion $dr_i^\alpha/dt = \varepsilon^{\alpha\beta}\sum_{j\neq i}^N \omega_j \frac{r^\beta - r_j^\beta}{|\mathbf{r}-\mathbf{r}_j|^2}[m|\mathbf{r} - \mathbf{r}_j|K_1(m|\mathbf{r} - \mathbf{r}_j|)]$, the factor in the square brackets is not singular in the origin and it only affects the spatial decay. The fact that the potential is pure gauge at infinity $(-\nabla\psi \times \hat{\mathbf{n}})^\alpha \sim g^{-1}dg$, with $g \in U(1)$, i.e., $g = \exp(i\xi)$, shows that it is a topological mapping between the circle at infinity on the plane and the group $U(1)$ and suggests to adopt for the gauge field the Chern-Simons Lagrangian density $\mathcal{L}_{\rm CS} = \frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma$, where $\varepsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor in $2+1$ dimensions [α, β, and γ can take three values: 0, 1, 2, corresponding to the time and the two coordinates (x, y)] and κ is a constant. It is known that this Lagrangian does not lead by itself to dynamical equations for the potential A_μ since it is first order in the time derivatives; it only represents a constraint on the dynamics, analogous to the Lorentz force in an external magnetic field. The Chern-Simons action is a generalization of the helicity: it is a topological quantity whose density is an exact differential form.

The matter field $\phi(x, y, t)$ must be complex since the vorticity carried by any pointlike vortex appears as a sort of electrical charge. The kinematical part of the matter field in the Lagrangian consists as usual in the squared momentum but with the covariant derivatives, to reflect the minimal coupling with the gauge field $\mathcal{L}_{\rm kin} = -(D^\mu\phi)^\dagger(D_\mu\phi)$, where $D_\mu = \partial_\mu + A_\mu$. The nonlinear self-interaction of the matter field is expressed as a potential $V(\phi)$ and must ensure, via classical Higgs mechanism, the generation of a finite mass m for the photon. If in addition we ask to support self-duality [14,15] then $V(\phi) \sim |\phi|^2(|\phi|^2 - v^2)^2$ (where v^2 is a constant).

We consider the case where all discrete vortices have the same absolute vorticity ω with two possible signs corresponding to positive and negative vorticity. Since the scalar field ϕ results from the density of positive and negative vortices, we note that the elementary vortices have much in common with complex Weyl spinors. It is then appropriate to work in the most general formulation, in which the fields ϕ and A_μ belong to the adjoint representation of the $SU(2)$ algebra. Then the Lagrangian density has the expression [16,17],

$$\mathcal{L} = -\kappa\varepsilon^{\mu\nu\rho}{\rm tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - {\rm tr}[(D^\mu\phi)^\dagger(D_\mu\phi)] - V(\phi, \phi^\dagger) \tag{3}$$

with

$$V(\phi, \phi^\dagger) = \frac{1}{4\kappa^2}{\rm tr}[([[\phi, \phi^\dagger], \phi] - v^2\phi)^\dagger([[\phi, \phi^\dagger], \phi] - v^2\phi)]. \tag{4}$$

This is the $(2+1)$-dimensional field-theoretical framework for the continuum limit of the CHM pointlike vortex model. Here $D_\mu = \partial_\mu + [A_\mu, \,]$ and the $\dagger$ means Hermitian conjugate. The Euler Lagrange equations are $D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^\dagger}$, $-\kappa\varepsilon^{\nu\mu\rho}F_{\mu\rho} = iJ^\nu$, where the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The current is $J^\mu = -i([\phi^\dagger, D^\mu\phi] - [(D^\mu\phi)^\dagger, \phi])$ and the Gauss law constraint is $-2\kappa F_{12} = iJ^0$ or $2\kappa F_{12} = [\phi^\dagger, D_0\phi] - [(D_0\phi)^\dagger, \phi]$. Very detailed calculations are presented in [18]. The action functional for the Lagrangian density (3) can be written in the Bogomol'nyi form [17], from which one derives that the extremum of the action is realized by the states verifying the self-duality equations

$$D_-\phi = 0 \quad F_{+-} = \frac{1}{\kappa^2}[v^2\phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger], \tag{5}$$

where $D_- \equiv D_1 - iD_2$, $A_\pm \equiv A_1 \pm iA_2$, and $F_{+-} \equiv \partial_+ A_- - \partial_- A_+ - [A_+, A_-]$.

As suggested in previous works [19,20], the following algebraic *ansatz* can be adopted

$$\phi = \phi_1 E_+ + \phi_2 E_-, \qquad \phi^\dagger = \phi_1^* E_- + \phi_2^* E_+,$$
$$A_+ = aH, \qquad A_- = -a^* H.$$

For this rank 1 algebra the Chevalley basis is $\{E_\pm, H\}$ with $[E_+, E_-] = H$, $[H, E_\pm] = \pm 2E_\pm$, ${\rm tr}(E_+E_-) = 1$, ${\rm tr}(H^2) = 2$. Using the *ansatz* and introducing the notations $\rho_1 \equiv |\phi_1|^2$, $\rho_2 \equiv |\phi_2|^2$ we obtain for the second self-duality equation

$$\frac{\partial a}{\partial x_+} + \frac{\partial a^*}{\partial x_-} = \frac{1}{k^2}(\rho_1 - \rho_2)[v^2 - 2(\rho_1 + \rho_2)]. \tag{6}$$

The first self-duality equation leads to

$$\frac{\partial\phi_1}{\partial x} - i\frac{\partial\phi_1}{\partial y} - 2\phi_1 a^* = 0$$
$$\frac{\partial\phi_2}{\partial x} - i\frac{\partial\phi_2}{\partial y} - \phi_2 a^* = 0 \tag{7}$$

and two analogous equations derived from the Hermitian conjugate of the first self-duality equation, $(D_-\phi)^\dagger = 0$.

From here we obtain $\Delta\ln\rho_1 = -\frac{1}{\kappa^2}(\rho_1 - \rho_2)[2(\rho_1 + \rho_2) - v^2]$ and $-\Delta\ln\rho_2 = -\frac{1}{\kappa^2}(\rho_1 - \rho_2)[2(\rho_1 + \rho_2) - v^2]$. The two equations are combined to give $\Delta\ln(\rho_1\rho_2) = 0$.

We adopt the simplest solution of the Laplace equation (see below) and choosing the constant of integration we have $\rho_1\rho_2 = v^4/(16p^2)$, with p a positive constant. Then using the normalizations $\rho \equiv \frac{\rho_1}{v^2/(4p)} = \frac{v^2/(4p)}{\rho_2}$ we write the equation for ρ_1

$$\Delta \ln\rho = -\frac{1}{4p^2}\left(\frac{v^2}{\kappa}\right)^2\left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - p\right]. \quad (8)$$

Making the substitution $\psi \equiv \ln\rho$ we obtain $2p^2(\frac{\kappa}{v^2})^2\Delta\psi + \sinh\psi(\cosh\psi - p) = 0$. The same equation is obtained starting from the equation for ρ_2, after a change of the unknown function, $\psi \to -\psi$. We scale the coordinates and obtain

$$\Delta\psi + \frac{1}{2p^2}\sinh\psi(\cosh\psi - p) = 0. \quad (9)$$

This is the equation governing the stationary asymptotic states of the CHM equation.

The physical meaning of the constant v^2/κ becomes clear when we investigate the far region on the plane. There the scalar function is approximately constant and close to the vacuum value, v^2. Then we obtain for the current $J^\mu \simeq 2iv^2A^\mu$. Using this expression in the second equation of motion $F_{\sigma\tau} = -\frac{i}{2\kappa}\varepsilon_{\mu\sigma\tau}J^\mu$ we obtain the equation $\partial_\tau\partial_\tau A_0 - (\frac{v^2}{\kappa})^2 A_0 = 0$. The solution of this equation is, in cylindrical geometry, $A_0(r) = K_0(mr)$. From here we conclude that the mass of the photon is

$$m = \frac{v^2}{\kappa} \quad (10)$$

and this mass is generated via the Higgs mechanism adapted to the Chern-Simons action (see review by Dunne in [20]). The photon acquires a mass because it moves in a background where the scalar field is equal with the vacuum value, v^2, which is not zero. This mass induces the short spatial range of the interaction in the discrete vortices model, introduced by Stewart and Morikawa in meteorology. We have $m = v^2/\kappa = 1/\rho_s$. In physical terms $\kappa \equiv c_s$, $v^2 \equiv \Omega_{ci}$. The vortical flows of the CHM equation are excitations over the background of "vorticity" represented by the Larmor gyration, intrinsically present in the CHM equation.

For different values of p, the differences are reflected in: the unit of space, ρ_s/p; the stream function ψ is shifted with $\ln p$; and the second factor of the nonlinear term in the equation may have both signs. The plot of the pair (ψ, ω) for $p = 10$ is shown in Fig. 1.

We can compare qualitatively with experimental results, obtained for the turbulence in fluids of geophysical interest [21]. We note that our Fig. 1 is very similar to the scatter plots presented in Figs. 21 of [21]. Comparison with numerical simulations of Seyler [22] show the close similarity between our graph (ψ, ω) with the figures presented in this reference.

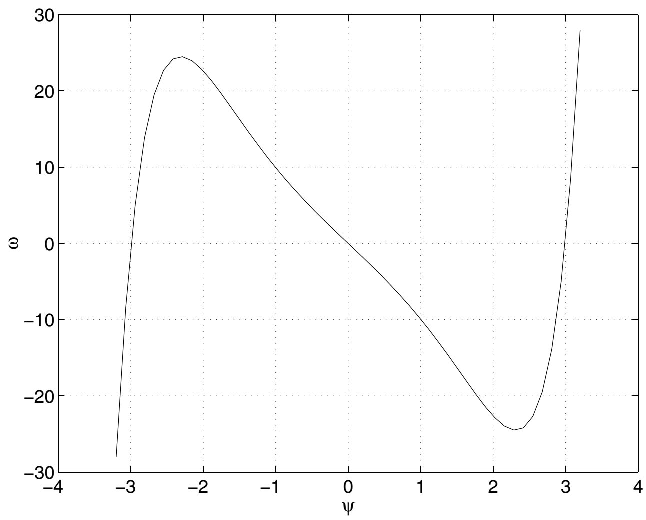

FIG. 1. Plot of the pair (ψ, ω).

The vortex solutions of the equation we have derived have the property that the vorticity $\omega = \Delta\psi$ and the amplitude of the density of the scalar function $\sim\phi\phi^\dagger$ must vanish at the same points, as shown by the second of the self-duality equations, Eq. (5). There are many situations where the vortex in atmosphere has a "doughnut" shape.

In general we have a family of differential equations parametrized by solutions of the Laplace equation, $\Delta\ln(\rho_1\rho_2) = 0$. In cylindrical geometry one can take $\rho_1\rho_2 = (H\theta + G)(C\ln r + D) + \Sigma_{n=1}^{\infty}(Ar^n + Br^{-n}) \times [E\cos(n\theta) + F\sin(n\theta)]$ ($A \ldots H$ constants). This might explain the scatter plots (see Fig. 5 of Ref. [23]) where a clear dependence of ω to ψ could not be identified.

The equation (9) describes systems defined by two elements: (1) a condensate of vorticity at infinity v^2 and (2) an intrinsic space scale ρ_s (or, equivalently, a finite sound speed, κ).

Since no auto-Backlund transform exists [24] this equation is probably not exactly integrable by the inverse scattering transform. The equation is difficult to integrate numerically, too. We have implemented the code GIANT [25] and have carried out a very large number (still insufficient) of numerical experiments. On rectangular space domains with dimensions from a fraction to few units of the fundamental length ρ_s we obtain systematically solutions of the symmetric vortex type. They cover several interesting physical systems. For the atmospheric vortex we obtain the morphology of the typhoon, with sharp decrease of the azimuthal velocity toward the center and much slower decay toward the periphery, Fig. 2. The vorticity is concentrated in the center and is almost zero in the rest. The diameter of the circle of maximum tangential wind (the *eye*) and the vorticity are in reasonable range but the velocity is higher than in observations, possibly due to the 2D geometry and the absence of viscosity in our model (see [26] for details). For the plasma vortex [27],

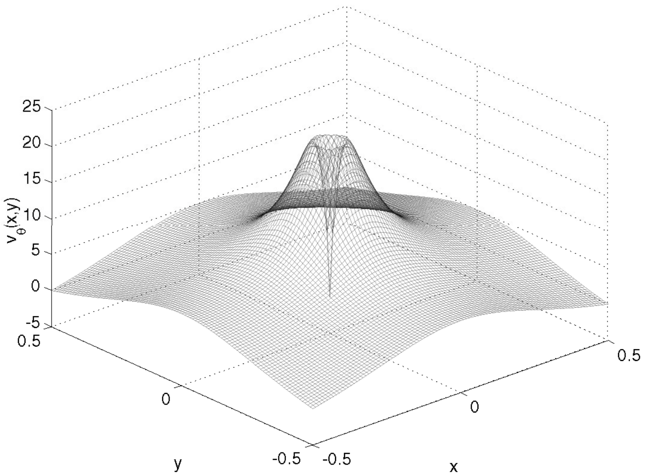

FIG. 2. The azimuthal velocity for a typical solution of Eq. (9) at $p = 1$. We use transparency to show the fast decrease of v_θ in the center.

after identifying an effective space scale $\rho_s \rightarrow \rho_{\text{eff}}$, the solution of Eq. (9) reproduces satisfactorily the experimental data for the vorticity and the azimuthal velocity profiles. We also obtain as approximative solutions sets of vortices with extremely narrow vorticity concentrations and almost zero on the rest of the space domain, similar to the crystal of vortices observed in non-neutral plasma. Here the condensate of vorticity is generated by the constant density of electric charge.

In conclusion, we have presented a field-theoretical framework for the pointlike vortices models of two-dimensional plasma and atmosphere. We have shown that the extremum of the action corresponds to stationary self-dual states and we have found the differential equation governing these states. The comparison with the experiment and numerical simulation is favorable.

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