Transition to Collisionless Ion-Temperature-Gradient-Driven Plasma Turbulence: A Dynamical Systems Approach

R. A. Kolesnikov and J. A. Krommes
Plasma Physics Laboratory, Princeton University, P.O. Box 451, Princeton, New Jersey 08543-0451, USA
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The transition to collisionless ion-temperature-gradient-driven plasma turbulence is considered by applying dynamical systems theory to a model with 10 degrees of freedom. The study of a four-dimensional center manifold predicts a ''Dimits shift'' of the threshold for turbulence due to the excitation of zonal flows and establishes (for the model) the exact value of that shift.

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Understanding the regulatory mechanisms of turbulent transport is key for understanding the magnetic confinement of plasmas. Nonlinearly generated $\boldsymbol{E} \times \boldsymbol{B}$ poloidal (zonal) flows (ZF's) play a central role in that process [1]. It is frequently said that ZF's shear apart eddies associated with the underlying turbulence, thus reducing the radial transport. In addition to being a topic of intense current research in the fusion context [2], ZF's are also important in geophysical contexts [3] and are related to shear-flow turbulence in neutral fluids [4].

An extreme example demonstrating the importance of ZF's is the so-called *Dimits shift* (here represented by Δ), which is a nonlinear upshift of the critical temperature gradient for the onset of ion-temperature-gradient-driven (ITG) plasma turbulence. Let that gradient be measured by ϵ, and let the threshold for linear instability be ϵ_c. According to large collisionless gyrokinetic [5] and gyrofluid [6] simulations of ITG systems slightly above marginal stability, there is a regime $\epsilon_c < \epsilon < \epsilon_*$ for some ϵ_* in which only ZF but no drift wave (DW) activity (and hence no radial transport) is observed. The Dimits shift is defined to be $\Delta \doteq \epsilon_* - \epsilon_c$ ($\doteq$ is used for definitions); ϵ_* is identified with the onset of DW turbulence. Rogers *et al.* [7] associated ϵ_* with tertiary modes that grow to nonlinear amplitudes and damp the ZF's. They discussed three stages as ϵ is increased: (i) primary instability of the DW's (collectively abbreviated by D); (ii) secondary instability of zonal modes Z (driven by D), which then (totally) suppress D; (iii) tertiary instability: Z is destabilized. Such nomenclature might suggest that one search for a sequence of three bifurcations occurring at $\epsilon^{(1)} \equiv \epsilon_c$, $\epsilon^{(2)}$, and $\epsilon^{(3)} \equiv \epsilon_*$. However, there is a fundamental difficulty with any steady-state scenario that relies on the loop $D \rightarrow Z \rightarrow D$ because one cannot close that loop with $D = 0$ but $Z \neq 0$. (Such a loop with nonzero values of both D and Z is considered in the statistical theory of fully developed DW-ZF turbulence [8].) Dastgeer *et al.* [6] seem to suggest that certain resonances enhance the ZF response, but still Z cannot be driven if $D \equiv 0$. No distinct $\epsilon^{(2)}$ is observed in the simulations. Instead, as Rogers *et al.* noted, ZF's are excited by a burst of DW's [through a Kelvin-Helmholtz (KH) instability of radial streamers], which then die away leaving only the ZF's.

A quantitatively accurate calculation of Δ in the face of complicated toroidal physics is formidable and is best left to large simulations. Although those are invaluable for the detailed modeling of complex behavior in real devices, they are cumbersome, expensive, and frequently ill suited for the identification and detailed understanding of basic conceptual issues. To focus on those and clarify some subtle asymptotic behavior, we here consider the opposite extreme and perform a dynamical systems analysis [9–11] of the ''simplest'' model of an electrostatic, collisionless (undamped ZF's [12]), curvature-driven ITG system near marginal stability. Unlike some other models of recent interest [13], our model captures a zonal-flow-driven Δ. Within the confines of the admittedly very simple model, we are able to calculate Δ exactly as a function of physical parameters. More importantly, we believe that the insights gained from this calculation remain relevant for more physically complete models.

A bifurcation is a change ''in the qualitative structure of the solutions'' [9] of a nonlinear system as a parameter such as ϵ varies. For many physicists, intuition about bifurcation phenomenology has been strongly influenced by the simplest normal forms [9]. For systems with linear waves, the two-dimensional (2D) Hopf bifurcation is especially relevant. If that pertained to the collisionless ITG problem and if it were supercritical [14], then slightly above linear threshold the DW's would saturate at a small amplitude $\propto (\epsilon - \epsilon_c)^{1/2}$; for an example, see the calculations of collisionally damped ZF's in Refs. [15,16]. However, that is not observed. If the bifurcation were subcritical [14], then the DW's would jump to a finite level as ϵ is increased beyond ϵ_c; that is not observed either.

In fact, the simple Hopf bifurcation does not apply to the strictly collisionless problem. A systematic way of proceeding is to exploit the center manifold theorem [9,10]. Let there be n_0 linear eigenvalues λ ($\partial_t \rightarrow e^{\lambda t}$) having zero real part, with all other eigenvalues having strictly negative real parts (this condition defines the bifurcation point in the space of parameters, e.g., ϵ). The theorem then states that the dynamics are attracted as $t \to \infty$ to a smooth n_0-dimensional invariant subspace, the *center manifold* (CM). Since an ITG model [17] must involve at least two coupled fields (usually vorticity ω and pressure P) in order that the system contains a self-consistent linear instability (the Hasegawa-Wakatani paradigm [14] is similar in this regard), the dimensionality of the CM is the sum of at least 2 (for the complex DW amplitude) plus 2 (for the two real undamped zonal fields); thus, the CM is at least 4D. The strong degeneracy associated with the presence of undamped modes and the consequently larger CM are responsible for the unusual dynamical behavior that underlies the Dimits shift. We demonstrate that explicitly for a simple model with 10 real degrees of freedom, both by perturbative construction of the CM (and qualitative analysis of the resulting dynamics) and by exact calculation of the relevant fixed point of the full nonlinearity. However, the qualitative insights transcend the model.

We are motivated by simplified gyrofluid ITG models that include magnetic curvature, such as discussed in Ref. [18]. We take $\boldsymbol{u} = (\omega, P)^T$ and associate the unit vectors $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{x}}$, and $\hat{\boldsymbol{y}}$ with the magnetic-field, radial, and (essentially) poloidal directions, respectively. The system (considered as 2D in the plane perpendicular to $\hat{\boldsymbol{z}}$) is $\partial_t \boldsymbol{u}(\boldsymbol{x}, t) = \hat{\mathsf{M}} \cdot \boldsymbol{u} + \hat{\boldsymbol{N}}(\boldsymbol{u}, \boldsymbol{u})$, with the hats denoting differential operators and the nonlinear term describing simple $\boldsymbol{E} \times \boldsymbol{B}$ advection: $\hat{\boldsymbol{N}}(\boldsymbol{u}, \boldsymbol{u}) = -\hat{z} \times \nabla\varphi \cdot \nabla\boldsymbol{u}$. The electrostatic potential φ follows from $\varphi = \hat{\mathcal{D}}^{-1}\omega$, where $\hat{\mathcal{D}} \doteq \hat{\alpha} - \hat{\nabla}^2$, $\hat{\alpha}$ being zero for convective cells ($k_\parallel = 0$) and the identity operator otherwise [8]. We take

$$\hat{\mathsf{M}} = \begin{pmatrix} -i(\hat{\Omega} - i\hat{\eta}) & -i\hat{b} \\ i\hat{\epsilon} & -\hat{d} \end{pmatrix}.$$

Here $\hat{\Omega} \doteq -2i(\hat{\mathcal{D}}^{-1} + \tau)\hat{\partial}_y$ (τ being the ratio of ion and electron temperatures) is associated with the DW frequency; $\hat{\eta} \doteq -\mu\hat{\nabla}^2$ describes collisional damping (on only the DW's); $\hat{b} \doteq -2i\hat{\partial}_y$; $\hat{\boldsymbol{\epsilon}} \doteq -i\tau\hat{\mathcal{D}}^{-1}[L_T^{-1} - (1 + \tau - \hat{\mathcal{D}})]\hat{\partial}_y$, where L_T is a normalized temperature-gradient scale length whose presence reflects magnetic curvature drive [19]; and $\hat{d} \doteq \hat{\bar{\nu}} + \hat{\eta}$, where $\hat{\bar{\nu}} = -\nu|\hat{\partial}_y|$ represents Landau damping [18].

We do not investigate even these simplified partial differential equations (PDE's) in full detail. In search of qualitative understanding about the basic phenomenology, we consider energetically self-consistent Galerkin truncations [11] in which the fields are represented as $\sum_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}(t) \times \sin(k_x x)e^{ik_y y}$. In choosing a standing wave in x, we subscribe to an argument from Ref. [15], which asserts that this crudely represents the localizing effect of magnetic shear. The lowest truncation retains $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$, where $1 \equiv (k_x, k_y)$, $2 \equiv (2k_x, 0)$, and $3 \equiv (3k_x, k_y)$. $\boldsymbol{u}_1$ represents both the bifurcating DW as well as a damped eigenmode; $\boldsymbol{u}_3$ is a DW sideband S; and $\boldsymbol{u}_2$ represents zonal variation Z, present as a result of the nonlinear interaction between D and S. This model does not retain streamers ($k_x = 0$), so it does not capture the KH mechanism of Ref. [7]; however, it does permit ZF's to be generated from a DW transient.

In the Fourier representation, the $\hat{\mathsf{M}}$ operator involves various $\boldsymbol{k}$-dependent coefficients Ω_i, η_i, b_i, ϵ_i, and d_i with $i = 1, 2, 3$; it possesses distinct left and right marginal eigenvectors $\boldsymbol{p}^i$ and $\boldsymbol{q}_i$. The bifurcation parameter is $\epsilon \equiv \epsilon_1$. (When $\eta_1 = 0$, the threshold for linear DW instability is $\epsilon_c = 0$ and the marginal eigenvalue is $-i\Omega_1$.) By definition of the collisionless problem (no linear zonal damping), η_2 is taken to vanish. η_1 and η_3 are unfolding parameters in the sense of Ref. [9]. The equations are

$$\dot{\omega}_1 = -i(\Omega_1 - i\eta_1)\omega_1 - ib_1P_1 + \frac{1}{2i}\left[\left(\frac{1}{\mathcal{D}_1} - \frac{1}{\mathcal{D}_2}\right)\omega_1\omega_2 - \left(\frac{1}{\mathcal{D}_3} - \frac{1}{\mathcal{D}_2}\right)\omega_3\omega_2\right], \quad (1a)$$

$$\dot{P}_1 = i\epsilon\omega_1 - d_1P_1 + \frac{1}{2i}\left[\left(\frac{\omega_1}{\mathcal{D}_1}P_2 - \frac{\omega_2}{\mathcal{D}_2}P_1\right) - \left(\frac{\omega_3}{\mathcal{D}_3}P_2 - \frac{\omega_2}{\mathcal{D}_2}P_3\right)\right], \quad (1b)$$

$$\dot{\omega}_3 = -i(\Omega_3 - i\eta_3)\omega_3 - ib_3P_3 - \frac{1}{2i}\left(\frac{1}{\mathcal{D}_1} - \frac{1}{\mathcal{D}_2}\right)\omega_1\omega_2, \quad (1c)$$

$$\dot{P}_3 = i\epsilon_3\omega_3 - d_3P_3 - \frac{1}{2i}\left(\frac{\omega_1}{\mathcal{D}_1}P_2 - \frac{\omega_2}{\mathcal{D}_2}P_1\right), \quad (1d)$$

$$\dot{\omega}_2 = \left(\frac{1}{\mathcal{D}_1} - \frac{1}{\mathcal{D}_3}\right)\mathrm{Im}(\omega_1\omega_3^*), \quad (1e)$$

$$\dot{P}_2 = \mathrm{Im}\left(\frac{\omega_1}{\mathcal{D}_1}P_3^* + \frac{\omega_3}{\mathcal{D}_3}P_1^*\right) - \mathrm{Im}\left(\frac{\omega_1}{\mathcal{D}_1}P_1^*\right). \quad (1f)$$

Here $\omega_2 \equiv z_\omega$ and $P_2 \equiv z_P$ are real. The nonlinear terms of this system conserve $\mathcal{W} \doteq |\omega_1|^2 + |\omega_3|^2 + \omega_2^2$ and $\mathcal{P} \doteq |P_1|^2 + |P_3|^2 + P_2^2$.

Numerical solutions of Eqs. (1) reveal that for $\epsilon < \epsilon_*$ (for some ϵ_*) and apparently all initial conditions (IC's), fluctuations eventually die away leaving only ZF's (as in the full simulations). For many IC's, the final state is unique; i.e., many trajectories are attracted to a stable, nontrivial fixed point (at $z = z_0$, with all other fields vanishing). (Some other IC's lead to final states dependent on the IC's.) For $\epsilon > \epsilon_*$, the model does not saturate in general. That is of little concern for a qualitative discussion of Δ; higher-order truncations [20] do saturate. (The value of ϵ_* depends on the order of truncation.)

All of this behavior can be explained qualitatively, and ϵ_* can be predicted quantitatively for the model, by a bifurcation analysis that involves the construction of the CM. As we noted, it is critical to realize that when the zonal components are undamped [$\eta_2 = 0$; see Eqs. (1e) and (1f)], the CM is 4D. To (locally) construct the CM, we substitute $\boldsymbol{u}_1 = D\boldsymbol{q}_1 + \boldsymbol{y}_1$, $\boldsymbol{u}_2 = \boldsymbol{z}$, and $\boldsymbol{u}_3 = \boldsymbol{y}_3$, where $\boldsymbol{p}^{i\dagger} \cdot \boldsymbol{y}_i = 0$. $\{D, \boldsymbol{z}\}$ serve as coordinates on the center eigenspace; the $\boldsymbol{y}$'s describe the nonlinear curvature of the CM with respect to that space. Symmetry considerations [11] dictate that $\boldsymbol{y}_i = \mathbf{W}_i \cdot \boldsymbol{z}D + \cdots$, where D and $\boldsymbol{z}$

are treated as small and the $\mathbf{W}_i$ are constant matrices to be determined. That may be accomplished [9,10] by equating the time derivative of the power-series expansion of $\boldsymbol{y}_i$ with the evolution equation that follows from the restriction of Eqs. (1) to the CM. In detail, we follow the projection method advocated by Kuznetsov [10], which does not require a preliminary linear diagonalization. To lowest order, we are led to the 4D system $\dot{D} = \Gamma(z)D$ and $\dot{z} = (-\boldsymbol{a}\epsilon + \mathsf{A}\cdot z)I$, where $I \doteq |D|^2$. This can immediately be reduced to a 3D system for I and z by writing $D = \rho e^{i\theta}$ and noting that the θ dependence decouples; one has $\dot{I} = 2\Gamma_r(z)I$, where $\Gamma_r \equiv \mathrm{Re}\Gamma$. The two-vector $\boldsymbol{a}$ and the 2×2 matrix A are known, and $\Gamma(z)$ is known through $O(z^2)$ [$\Gamma_r(\mathbf{0})$ is the linear DW growth rate]. All elements of A are positive; both of its eigenvalues are negative. The supporting algebra and formulas for these quantities and others in the subsequent analysis will be displayed elsewhere.

For this reduced dynamics, not only is the origin $\mathcal{O}$ ($z = \mathbf{0}$, $I = 0$) a fixed point (linearly unstable for $\epsilon > \epsilon_c$), the entire $I = 0$ plane is invariant. This unusual behavior is the first indication that for the collisionless problem the origin does not have the same preferred status as in other, more conventional situations [15,16]. Indeed, the system also admits a nontrivial fixed point $\mathcal{F}$ at $I_0 = 0$ and $z_0 = \epsilon\mathsf{A}^{-1}\cdot\boldsymbol{a}$. The stability of $\mathcal{F}$ determines Δ.

One may perform a phase-plane analysis by noting that I cancels out under $\dot{z}_P/\dot{z}_\omega = dz_P/dz_\omega \equiv v(z)/u(z)$; see Fig. 1. All qualitative properties of that figure can be determined analytically. $\mathcal{F}$ is attracting in the z plane (for all ϵ); it passes through the origin as ϵ passes through 0. In the submarginal region $\epsilon < \epsilon_c$, $\Gamma_r(z) < 0$ for all sufficiently small z's. Thus all trajectories starting close to the origin are attracted to the $I = 0$ plane and end up close to the initial starting point (the final z may be either $\mathcal{F}$ or may depend on IC's [21]).

In the supermarginal region $\epsilon_c < \epsilon < \epsilon_*$, $\Gamma_r(z) > 0$ in the vicinity of $z = \mathbf{0}$ but is negative in the vicinity of $\mathcal{F}$. Then most trajectories starting close to $\mathcal{O}$ initially move away from it; they end up either at $\mathcal{F}$ [for sufficiently large $I(t = 0)$] or on the $I = 0$ plane at positions depending on IC's [21]. Such dynamics are consistent with the observed behavior above marginality: an initial burst of unstable DW's generates ZF's, which then annihilate the DW's leaving only a steady ZF component as $t \to \infty$. This generation (secondary instability) or annihilation process is *transient,* so it does not involve a distinct bifurcation point $\epsilon^{(2)}$. Also, note that, although $\mathcal{O}$ changes stability at $\epsilon = \epsilon_c$, $\mathcal{F}$ does not. Thus, ϵ_c does not serve as a distinct bifurcation point $\epsilon^{(1)}$ for the global system dynamics.

As ϵ is increased further through some ϵ_*, $\Gamma_r(z_0)$ becomes positive, many IC's are repelled from the $I = 0$ plane, and the system cannot saturate. (Simulations verify that higher truncations do saturate with nonzero levels of DW activity and characteristic chaotic behavior.)

Perturbative CM calculations provide only approximations to $\mathcal{F}$ and ϵ_*, and they cannot address the global

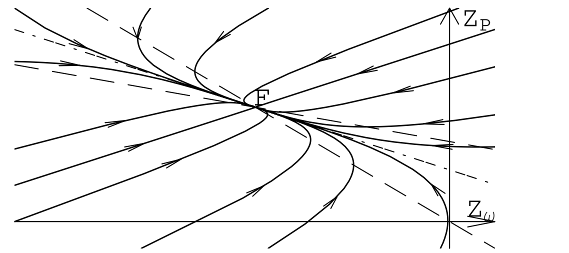

FIG. 1. Phase trajectories in the z plane, showing the nontrivial fixed point $\mathcal{F}$ (stable in the z plane) the overall stability of which determines the Dimits shift. Upper dashed line: $v(z) = 0$ (slope $dz_P/dz_\omega = 0$); lower dashed line: $u(z) = 0$ (slope $= \infty$); dash-dotted lines: eigenvector directions (one such line is obscured by the trajectory at approximately 45°).

structure of the phase space. Fortunately, the present model is simple enough that certain quantities can be calculated exactly. Rigorous equations for z_0 are motivated by the observation that dynamics ought to relax rapidly to the CM. Since ω_1 has a component in the CM, we define $P_1' \doteq P_1/\omega_1$, $\omega_3' \doteq \omega_3/\omega_1$, and $P_3' \doteq P_3/\omega_1$. Although for $\epsilon < \epsilon_*$ all original variables (except for z) are dynamically driven to zero, the primed variables remain nonzero as $t \to \infty$. This expedites tracking the fixed point $z_0(\epsilon)$. Upon deriving evolution equations for the new variables from Eqs. (1), passing to a polar representation, and requiring that the primed amplitudes and phases be steady, we are led after tedious algebra to tractable equations for the position of $\mathcal{F}$. Further nontrivial algebra shows that to lowest order in ϵ the prediction agrees with that found from the perturbative CM construction. For any ϵ, numerical solution of the fixed-point equations demonstrates agreement with the numerically observed z_0 through six decimal places.

In principle, this nonperturbative calculation captures all (possibly global) fixed points of the original system (the perturbative CM calculation is local); however, we have found only the z_0 described previously. We have no proof that no other fixed points exist, although no other stable ones have emerged from an incomplete numerical search of the phase space. We believe that if they do exist they are all saddle points, which would not modify the qualitative asymptotics we have described [22].

With nonperturbative results in hand, we can formulate an exact equation for Δ [22]. We write $\omega_1 = \rho_1 e^{i\theta_1}$, divide Eq. (1a) by ω_1, and take the real part, obtaining (at z_0) $\dot{\rho}_1/\rho_1 = -\eta_1 + b_1 Y_1'$, where $Y_1' \doteq \mathrm{Im}\,P_1'$. [Nonlinear terms do not contribute due to the steady-state condition $\mathrm{Im}\,\omega_3' = 0$, which follows from Eq. (1e).] $\mathcal{F}$ is therefore destabilized in the I direction when $Y_1'(\epsilon_*) = \eta_1/b_1$. An exact formula for $Y_1'(\epsilon)$ is available, and we find agreement with our simulation value of ϵ_*. The destabilization process is not a KH instability but rather an ITG instability modified by stabilizing ZF shear.

An important observation emerges by considering the limit of small DW collisional dissipation $\eta_{1,3} \to 0$, for which it can be shown that $\epsilon_* \propto \eta_1/(\eta_3 - \eta_1)$. This ratio remains nonzero in the limit. Thus a nonzero Δ can arise even for vanishing collisional dissipation, as in the full collisionless simulations.

Now consider the addition of very weak zonal damping: $\dot{z} = -\eta_z z + \cdots$ ($\eta_z \equiv \eta_2$). This introduces a new, very long time scale and slightly perturbs the position of $\mathcal{F}$. For $\epsilon < \epsilon_*$, arbitrary IC's typically move rapidly to the vicinity of the original F, then slowly relax to the steady state. This disparity of time scales underlies the bursting behavior observed in, e.g., Refs. [23,24] for weakly collisional runs. That does not occur in the lowest-order truncation studied here, but does occur in higher-order ones [20], whose additional degrees of freedom allow $\mathcal{F}$ to be destabilized in other directions and thus the trajectories to be ejected from its vicinity after the slow relaxation. Preliminary long-time (many-burst) integrations of such truncations show relaxation to a quasiregular state; limited computational resources precluded the authors of Ref. [23] from integrating more than a few bursts.

For sufficiently large η_z, the zonal modes are strongly stable and should no longer be used as coordinates on the CM, which is then 2D. For this case, a standard Hopf bifurcation occurs; the (straightforward) details will be presented elsewhere, as will a discussion of the modulational instability described by the associated Ginzburg-Landau equation. The radical differences in behavior between the undamped (weakly damped in reality) and strongly damped limits arises because of the interchange of the limits $t \to \infty$ and $\eta_z \to 0$ [25]. The signature of that interchange is the differing dimensionality of the CM's for the two cases. The Dimits shift occurs when the limit $\eta_z \to 0$ is taken first. This interchange of limits is one of our major points; it is not restricted to our specific model.

In summary, we have considered a very simple yet instructive model for the transition to collisionless ion-temperature-gradient-driven plasma turbulence. The excitation of zonal flows, important in various physics contexts, plays a crucial role in the dynamics of that transition. Here, by using tools from dynamical systems theory (especially the calculation of a local center manifold [26]), we have shown how the nonlinear upshift of the critical temperature gradient for the onset of turbulence (known as the Dimits shift Δ) is related to a certain fixed point and how Δ can be calculated in terms of the physical parameters of the model. We do not claim to have calculated Δ for a realistic system of PDE's, but we stress that the asymptotic interchange of limits related to very weak zonal damping underlies the very existence of Δ. We argue that dynamical systems analysis of nonlinear models possessing relatively small numbers of degrees of freedom is an instructive alternative to large simulations, particularly near the onset of turbulence. Both approaches have their places in the quest to understand the transport properties of magnetically confined plasmas and related nonlinear systems.

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