Designer Gravity and Field Theory Effective Potentials

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Motivated by the anti–de Sitter conformal field theory correspondence, we show that there is remarkable agreement between static supergravity solutions and extrema of a field theory potential. For essentially any function $\mathcal{V}(\alpha)$ there are boundary conditions in anti—de Sitter space so that gravitational solitons exist precisely at the extrema of $\mathcal{V}$ and have masses given by the value of $\mathcal{V}$ at these extrema. Based on this, we propose new positive energy conjectures. On the field theory side, each function $\mathcal{V}$ can be interpreted as the effective potential for a certain operator in the dual field theory.

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Introduction.—There is a long history of studying gravitational theories with anti–de Sitter (AdS) boundary conditions (see, e.g., [1]), and in recent years this has led to breakthroughs in string theory and models of extra dimensions [2,3]. We describe below a novel feature of these theories with AdS boundary conditions: For the same action, there can be many possible boundary conditions, and changing the boundary conditions changes the properties of the theory. In particular, we will see that one can ''preorder'' the number and masses of solitons in some theories—there are boundary conditions which yield any desired result. For this reason, we call this phenomenon ''designer gravity.''

Among the theories of gravity for which this is possible are certain supergravity theories. In fact, although this result is independent of string theory (and does not use supersymmetry), it was discovered while investigating the AdS conformal field theory (AdS-CFT) correspondence [2]. Furthermore, in cases where there is a field theory dual, the gravitational solitons can be used to compute certain effective potentials in the field theory.

We will consider theories of gravity coupled to a scalar field with potential $V(\phi)$. We require that V has a negative maximum, so that AdS is a solution and small scalar fluctuations are tachyonic, $m^2 < 0$. It has long been known that tachyonic scalars in $d+1$ dimensional AdS spacetime are stable provided their mass is above the Breitenlohner-Freedman (BF) bound [4] $m^2_{\rm BF} = -d^2/4$ (in units of the AdS radius). It has been shown much more recently that if

$$m^2_{\rm BF} \leq m^2 < m^2_{\rm BF} + 1, \tag{1}$$

then more general boundary conditions are possible which still admit a conserved finite total energy and preserve all the AdS symmetries [5,6].

For definiteness, we will focus on the case of $\mathcal{N} = 8$ gauged supergravity in four dimensions, and comment on generalizations at the end. This theory can be consistently truncated to include just gravity and a single scalar field with potential [7]

$$V(\phi) = -2 - \cosh(\sqrt{2}\phi), \tag{2}$$

so, setting $8\pi G = 1$, our action is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 + 2 + \cosh(\sqrt{2}\phi)\right]. \tag{3}$$

The potential (2) has a maximum at $\phi = 0$ corresponding to an AdS_4 solution with unit radius. It is unbounded from below, but small fluctuations have $m^2 = -2$, which is above the BF bound, and satisfies (1). (This mass corresponds to conformal coupling.)

In all asymptotically AdS solutions, the scalar ϕ decays at large radius as

$$\phi(r) = \frac{\alpha}{r} + \frac{\beta}{r^2}, \tag{4}$$

where r is an asymptotic area coordinate, and α and β can depend on the other coordinates. The standard boundary conditions correspond to either $\alpha = 0$ or $\beta = 0$ [4,8]. It was shown in [5] that $\beta = k\alpha^2$ (with k an arbitrary constant) was another possible boundary condition that preserves all the asymptotic AdS symmetries. We now consider even more general boundary conditions $\beta = \beta(\alpha)$. Although these will generically break some of the asymptotic AdS symmetries, they are invariant under global time translations. Hence there is still a conserved total energy, as we now show.

As discussed in [5], the usual definition of energy in AdS diverges whenever $\alpha \neq 0$. This is because the backreaction of the scalar field causes certain metric components to fall off slower than usual. The complete set of boundary conditions can be found in [5], but the main change is in g_{rr}:

$$g_{rr} = \frac{1}{r^2} - \frac{(1 + \alpha^2/2)}{r^4} + O(1/r^5). \tag{5}$$

The expression for the conserved mass depends on the asymptotic behavior of the fields and is defined as follows. Let ξ^μ be a timelike vector which asymptotically approaches a (global) time translation in AdS. The Hamiltonian takes the form

$$H = \int_\Sigma \xi^\mu C_\mu + \text{surface terms}, \tag{6}$$

where Σ is a spacelike surface, C_μ are the usual constraints, and the surface terms should be chosen so that the variation of the Hamiltonian is well defined. The variation of the usual gravitational surface term is given by

$$\delta Q_G[\xi] = \frac{1}{2}\oint dS_i \bar{G}^{ijkl}(\xi^\perp \bar{D}_j \delta h_{kl} - \delta h_{kl} \bar{D}_j \xi^\perp), \tag{7}$$

where $G^{ijkl} = \frac{1}{2} g^{1/2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})$, $h_{ij} = g_{ij} - \bar{g}_{ij}$ is the deviation from the spatial metric $\bar{g}_{ij}$ of pure AdS, $\bar{D}_i$ denotes covariant differentiation with respect to $\bar{g}_{ij}$, and $\xi^\perp = \xi \cdot n$, with n the unit normal to Σ. Since our scalar field is falling off more slowly than usual if $\alpha \neq 0$, there is an additional scalar contribution to the surface terms. Its variation is simply

$$\delta Q_\phi[\xi] = -\oint \xi^\perp \delta\phi D_i \phi dS^i. \tag{8}$$

Using the asymptotic behavior (4), this becomes

$$\delta Q_\phi[\xi] = r\oint (\alpha\delta\alpha) d\Omega + \oint [\delta(\alpha\beta) + \beta\delta\alpha] d\Omega. \tag{9}$$

Since there is a term proportional to the radius of the sphere, this scalar surface term diverges. However, this divergence is exactly canceled by the divergence of the usual gravitational surface term (7). The total charge can therefore be integrated, yielding

$$Q[\xi] = Q_G[\xi] + r\oint \frac{\alpha^2}{2} d\Omega + \oint [\alpha\beta + W(\alpha)] d\Omega, \tag{10}$$

where we have defined

$$W(\alpha) = \int_0^\alpha \beta(\tilde{\alpha}) d\tilde{\alpha}. \tag{11}$$

In addition to canceling the divergence in (10), the gravitational surface term contributes a finite amount M_0. For the spherically symmetric solutions we consider below, this is just the coefficient of the $1/r^5$ term in g_{rr}. Since α and β are now independent of angles, the total mass becomes

$$M = 4\pi(M_0 + \alpha\beta + W). \tag{12}$$

(For $\beta = k\alpha^2$, this agrees with the expression for the mass given in [5].)

Gravitational solitons.—We want to study solitons in this theory. These are nonsingular, static, spherically symmetric solutions. Writing the metric as

$$ds^2 = -h(r)e^{-2\chi(r)}dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega, \tag{13}$$

the field equations read

$$h\phi_{,rr} + \left(\frac{2h}{r} + \frac{r}{2}\phi_{,r}^2 h + h_{,r}\right)\phi_{,r} = V_{,\phi}, \tag{14}$$

$$1 - h - rh_{,r} - \frac{r^2}{2}\phi_{,r}^2 h = r^2 V(\phi), \tag{15}$$

$$\chi_{,r} = -\frac{r\phi_{,r}^2}{2}. \tag{16}$$

Regularity at the origin requires $h = 1$ and $h' = \phi' = \chi' = 0$. Rescaling t shifts χ by a constant, so its value at the origin is arbitrary. Thus solutions can be labeled by the value of ϕ at the origin. For each $\phi(0)$, one can integrate these ordinary differential equations and get a soliton. Asymptotically, ϕ behaves as (4), so we get a point in the (α, β) plane. Repeating for all $\phi(0)$ yields a curve $\beta_s(\alpha)$ where the subscript indicates this is associated with solitons. This curve is plotted in Fig. 1. [Since the potential $V(\phi)$ is even, it suffices to consider positive $\phi(0)$, which corresponds to positive α.] Note that solitons exist for arbitrarily small α. When $\alpha \ll 1$, $\phi(r)$ is small everywhere, and one might have thought a linearized approximation should be valid, implying no solitons could exist. This is incorrect, since for any $\alpha \neq 0$ the backreaction is always large asymptotically as shown in (5). Given a choice of boundary condition $\beta(\alpha)$, the allowed solitons are simply given by points where the soliton curve intersects the boundary condition curve: $\beta_s(\alpha) = \beta(\alpha)$.

We can now state our prescription for choosing boundary conditions to reproduce any prescribed set of solitons. Set

$$W_0(\alpha) = -\int_0^\alpha \beta_s(\alpha). \tag{17}$$

This function is universal, in the sense that it is independent of our choice of boundary conditions. Now given any smooth function $\mathcal{V}(\alpha)$ with $\mathcal{V}(0) = 0$, we write $\mathcal{V} = W_0 + W$ and take our boundary conditions to be $\beta = W'(\alpha)$. It follows immediately that the extrema of $\mathcal{V}$ are in one-to-one correspondence with solitons that obey these boundary conditions:

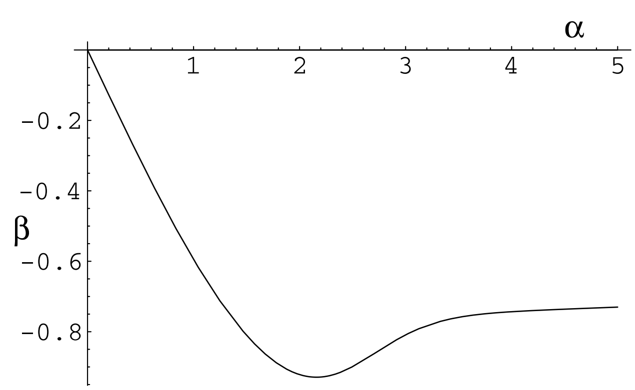

FIG. 1. The function β_s obtained from the solitons.

$$0 = \mathcal{V}' = W_0' + W' = -\beta_s + \beta. \quad (18)$$

So the extrema of $\mathcal{V}$ are precisely the points where $\beta_s = \beta$. Furthermore, the mass of each soliton is given by the value of $\mathcal{V}$ at the corresponding extremum. To see this, remember that static solutions are extrema of the mass [9]. Suppose we choose our boundary condition to be $\beta = \beta_s(\alpha)$. For this special case, all the solitons are allowed by the boundary conditions. Since we have a one parameter family of static solutions, the mass must be constant; i.e., all the solitons have the same mass. But this includes $\beta = \alpha = 0$, which is just AdS and has zero mass. So all the solitons have zero mass. From (12), with boundary conditions $\beta = \beta_s(\alpha)$, we have

$$0 = M_0 + \alpha\beta_s - W_0. \quad (19)$$

Therefore, for our general boundary condition $\beta = W'(\alpha)$, we have

$$M = 4\pi(M_0 + \alpha\beta + W) = 4\pi(W_0 + W) = \oint \mathcal{V} d\Omega, \quad (20)$$

where we have used the fact that $\beta = \beta_s(\alpha)$ for a soliton. Thus the mass of the soliton is indeed given by the value of $\mathcal{V}$ at the corresponding extremum. Notice that the only restriction on $\mathcal{V}$ [that $\mathcal{V}(0) = 0$] comes from the fact that we want the total mass of pure AdS to be zero.

We have also studied the stability of these solitons. The most likely mode to go unstable is a spherically symmetric scalar perturbation such as the one studied for hairy black holes in [10]. We have found numerically that this mode is indeed unstable if $\mathcal{V}'' < 0$. We expect the solitons with $\mathcal{V}'' > 0$ to be stable. This leads to a new class of "positive" energy conjectures [11]. For given boundary conditions, the minimum energy solution is expected to be static and hence one of the solitons we have been discussing. If $\mathcal{V}$ has a global minimum, then it seems likely that the energy of any supergravity solution cannot be less than the minimum mass soliton. Hence we are led to the following conjecture: *Given any smooth function $\mathcal{V}(\alpha)$ with $\mathcal{V}(0) = 0$ and a global minimum $\mathcal{V}_{\min}$, consider solutions to* (3) *with boundary condition $\beta = W'$, where $W = \mathcal{V} - W_0$ and W_0 is given by* (17). *Then the conserved mass of any nonsingular initial data set is bounded below by $4\pi\mathcal{V}_{\min}$.*

Field theory.—We now turn to the dual field theory interpretation. String theory on spacetimes which asymptotically approach $AdS_4 \times S^7$ is dual to the $2 + 1$ CFT describing the low energy excitations of a stack of M two-branes. This theory is not well understood, but we can learn something nontrivial using the gravitational solitons. With $\beta = 0$ boundary conditions, the bulk scalar ϕ is dual to a dimension one operator $\mathcal{O}$. One way of obtaining this CFT is by starting with the field theory on a stack of D two-branes and taking the infrared limit. In that description [12],

$$\mathcal{O} = TrT_{ij}\varphi^i\varphi^j, \quad (21)$$

where T_{ij} is symmetric and traceless, and φ^i are the adjoint scalars.

Let S_0 denote the CFT Lagrangian and consider the deformation

$$S = S_0 - k\int \mathcal{O}. \quad (22)$$

Using the standard AdS-CFT dictionary, the vacuum expectation value of $\mathcal{O}$ in this deformed theory is obtained by finding nonsingular static supergravity solutions with $\beta = -k$, but these are precisely our solitons. Given a soliton with $\beta = -k$, one has $\langle\mathcal{O}\rangle = \alpha$. Hence the function $\mathcal{V}(\alpha) = W_0 - k\alpha$ can be interpreted as the effective potential for $\langle\mathcal{O}\rangle$, where W_0 is the function (17) computed earlier from the soliton solutions. From Fig. 1 we see that there are three qualitatively different regions. For small k, there is a unique soliton and hence a unique nonzero value for $\langle\mathcal{O}\rangle$. For intermediate values of k there are two solitons, indicating there are two vacua, and for large k there are no solitons, indicating there is no vacuum at all.

Since our CFT lives on $S^2 \times R$ (it is dual to a bulk theory which approaches global AdS asymptotically), one might have expected a mass term $\frac{1}{2}m^2\varphi^2$ coming from the conformal coupling of the scalars to the curvature of the S^2. The radius of the S^2 is equal to the AdS radius, so one expects $m^2 = R/8 = 1/4$ in AdS units. Since $\mathcal{O}$ is quadratic in φ, the presence of a mass term would mean that for small $k < m^2/2$ the vacuum would be unchanged and $\langle\mathcal{O}\rangle = 0$. However, this is not what we find. Figure 1 shows that β_s is linear in α for small α, so that W_0 is quadratic in α; hence the vacuum expectation value $\langle\mathcal{O}\rangle$ is shifted even for small k. This is illustrated in Fig. 2 for $k = 1/2$. For slightly larger k a new maximum appears at larger α and the theory becomes nonperturbatively unstable.

Now suppose we replace $-k\int\mathcal{O}$ in (22) with $\int W(\mathcal{O})$, where W is an arbitrary function of $\mathcal{O}$. Remarkably, the expectation values $\langle\mathcal{O}\rangle$ in different vacua are again given by the extrema of $\mathcal{V} = W_0 + W$, where W_0 is the same function as above, and W is unchanged. This is because the

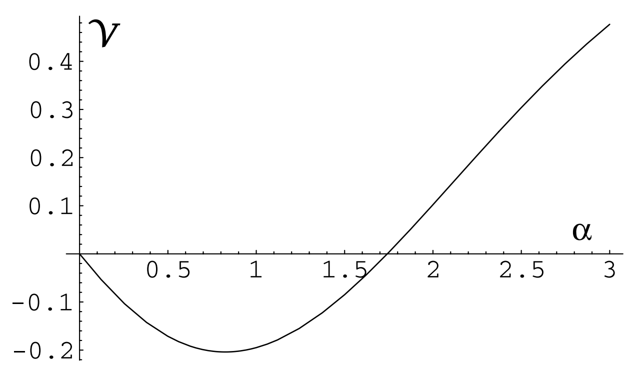

FIG. 2. The effective potential $\mathcal{V}(\alpha) = W_0 - \frac{1}{2}\alpha$.

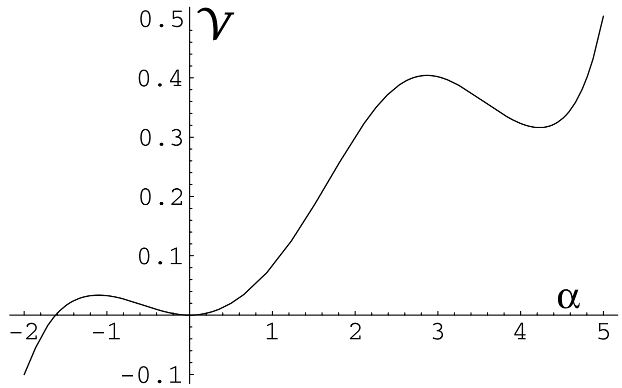

FIG. 3. The effective potential $\mathcal{V} = W_0 - \frac{1}{4}\alpha^2 + \frac{1}{40}\alpha^3$.

addition of $\int W(\mathcal{O})$ to the CFT action corresponds in the bulk to using the modified boundary conditions $\beta = W'$ [13], but we have already seen that the extrema of $\mathcal{V}$ correspond to solitons with precisely these boundary conditions. The fact that the function W does not receive any corrections in the effective potential is surprising and reminiscent of a nonrenormalization theorem, but we are dealing with configurations that are far from supersymmetric. Perhaps it is related to taking the large N limit or to properties of the operators that are dual to scalars with masses in the range (1). An example of the effective potential in the presence of a multitrace deformation W that yields a nontrivial false vacuum is given in Fig. 3.

Discussion.—In summary, we have seen that one can preorder solitons in supergravity in the following sense: Given essentially any function $\mathcal{V}(\alpha)$, there are boundary conditions such that gravitational solitons exist precisely for each extremum of $\mathcal{V}(\alpha)$ and have masses given by the value of $\mathcal{V}$ at the corresponding extremum. Furthermore, in supergravity theories with a field theory dual, the function $\mathcal{V}$ can be interpreted as the effective potential for the dual operator $\mathcal{O}$. It would be interesting to perform an independent field theory calculation of the effective potential, for instance, in the case of a simple single trace deformation, since this would provide a new test of AdS-CFT. One can also extend our results from solitons to black holes with scalar hair. One can preorder black holes either in terms of their size or temperature. In the latter case, a similar bulk analysis yields again a function $\tilde{\mathcal{V}}(\alpha)$ that can be interpreted as the finite temperature effective potential for $\mathcal{O}$ in the dual field theory. This will be discussed in more detail in [14].

Although we have focused on a scalar field with $m^2 = -2$ in four-dimensional $\mathcal{N} = 8$ supergravity, the gravity side of the story can be generalized to other dimensions and all scalars with masses in the range (1). For asymptotically AdS_{d+1} spacetimes, a scalar field with mass m asymptotically falls off as

$$\phi = \frac{\alpha}{r^{\Delta_-}} + \frac{\beta}{r^{\Delta_+}}, \tag{23}$$

where

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2}. \tag{24}$$

If the mass is in the range $m^2_{\mathrm{BF}} \leq m^2 < m^2_{\mathrm{BF}} + 1$, then a finite, conserved total energy can be defined for any boundary condition $\beta(\alpha)$. The variation of the scalar surface term is still (8), so inserting this asymptotic behavior of ϕ yields

$$M = \mathrm{Vol}(S^{d-1})\left[\frac{d-1}{2}M_0 + \Delta_-\alpha\beta + (\Delta_+ - \Delta_-)W\right]. \tag{25}$$

One can again construct the soliton curve $\beta_s(\alpha)$ and find boundary conditions that admit any desired soliton solutions. If a dual field theory exists, then one can again compute effective potentials for the operators dual to the bulk scalar field.

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